THE
MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., AND PROF. E. T. WHITTAKER, M.A., F.R.S.

LONDON:
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

VOL. VII. MARCH, 1914. No. 110.

The Mathematical Association.

ANNUAL MEETING, 1914.

The Annual Meeting was held at the London Day Training College, Southampton Row, London, W.C., on 7th January, 1914, the President, SIR GEORGE GREENHILL, F.R.S., in the Chair.

PRESIDENTIAL ADDRESS.

THE USE OF MATHEMATICS.

THE use of Mathematics we understand as not merely utilitarian applications, such as those by which astronomy is employed in navigation, "whereby our nautical logbooks can be better kept, and water transport of all kinds has grown more commodious"; flight too can be made in the air; but also the cultivation at the same time of the intellect, to think correctly, and arrive at the right conclusion from data which are certain; and the enlargement generally of the interest in this life.

And so our subject received its name from MAΘHMATA, meaning disciplined exact knowledge; and formed an important branch in the old University ideal, to teach how to live a life, not merely to make a living. To keep the soul alive, body and soul together, and to cure intellectual timidity, the worst form of cowardice.

The country will never forget the fright and despair of the last century, when our military decrepitude was dragged to the light —decrepitude the result of thirty years' work of a scheme, organising every detail to a War Office satisfaction—everything, except the one essential, Victory.

The scheme made a bid for popularity by the abolition of mathematics, as being of no use to the General in the direction of military operations on a large scale. How Napoleon would smile if he could hear this!

So the excellent services of the Reverend Sir John Twisden were cancelled at the Staff College, and our Staff Officer and General went out to Africa with none of the disciplined mathematical habit, as a cure for intellectual timidity.

Hence the origin of so much called regrettable.

The official history of the war gives an imperfect catalogue, and we must confine ourself to a single illustrative example.

When the war was declared at an end by official proclamation (it really lasted another year), the General was ordered to cut all his losses, and scuttle home as fast as possible. But the staff could not count. It was not disciplined enough to calculate the time required to repatriate three or four hundred thousand men over six thousand miles.

Someone guessed a fortnight, instead of a year at least, actually eighteen months; and all stores were ordered to be sold off except a fortnight's supply. But before the fortnight was up, the stores came rolling again into Johannesburg in railway trucks, bought back at a highly enhanced price. Hence a War Store Commission to inquire into the loss of a few of the millions.

So, although mathematics may be no use to a General in the routine of peace, and little of any other knowledge he will tell us, the case is altered when he goes to war.

Then it is very important to the taxpayer for the general staff to have some of the trained discipline of mathematical thought, and a knowledge of space, time, and pace, to prevent the bill mounting up by the hundred million. But so far the only effect of the sharp lesson is seen in a class of staff officers attached to a school in Clare Market to learn—book-keeping.

Looking abroad such a short distance, as into France, what a contrast can be seen! The French army, by superior zeal and disciplined trained intelligence, without recourse to its numbers, can blow our best effort to atoms. The country is not safe.

In the mediæval division of science into Philosophy, Moral and Natural, Mathematics was classed under natural philosophy, the old name of mathematical reasoning, with experiment, on the laws of nature. Natural Science was called Natural History.

But the new Physics, which looms so large in a modern scheme of education, has come to mean—natural philosophy with the mathematics left out.

And new mathematics allies itself with moral philosophy, and condemns the old natural philosophy as wanting in rigour—

principia omni rigore carentia—like Newton's, abolished in the latest scheme at Cambridge, so as to give more time to the new sermon in mathematics, in spite of Herschel's deprecation—

ΜΗ ΓΕΝΟΙΤΟ.

The mere mathematician asks, in the mathematics he calls advanced, for no more apparatus than a lump of chalk (his calculus)—"*voici mon sceptre*," he says, like Marat in the play of Charlotte Corday, holding up a pen!

If he would add a length of thread for use on the blackboard with his lump of chalk, he would have all the instrumental appliance postulated by Euclid in his Elements. No need for a box of instruments. The thread here is out of doors the gardener's string; the box of instruments would be useless to him. Or else the thread turns to the rope of the rope-stretcher in the vocational geometry of Land Surveying, or his chain, divided into links.

Stretch the thread so as to lie evenly between the two ends; the test of evenly is the optical test of looking along the thread. Free in the air the thread cannot be made to lie evenly between any two points, however hard it is stretched; it will break first, unless vertical. But lay it along the blackboard placed horizontal: chalk the thread, and twang it on the board as the carpenter does, and a straight chalk line is the result.

Euclid's test can be made incidentally of the blackboard as a plane superficies—that is, in which any two points being taken, the straight line between them lies wholly in that superficies—verified again by the optical test.

In Whitworth's method the planes were made in triplets to reveal in each other the "defect of perfection."

The same piece of thread can serve as the radius of a circle described on the board, with one thumb over the centre and the lump of chalk—between finger and thumb at the other end—to draw the circumference.

Fold a sheet of paper and a straight line is made. The sheet of paper is the physical representation of the plane superficies taken off a blackboard, tested beforehand for flatness by the thread. Fold the paper again and the right angle appears, and the square; and so on for the familiar exercises in paper folding, for making a pentagon and hexagon.

How did the division arise of the clock-face, into hours and minutes? as old as history can go back in Assyrian and Chaldæan record. It is only another version of Euclid's problem to inscribe a regular quindecagon in a circle. The side is four minutes on the clock, one minute behind I o'clock, starting from XII.

The subsequent division of the minute into six degrees is a Chaldæan mystery.

This is for space of three dimensions, the finest space we ever were in. But the absurdity is revealed of the metric decimal subdivision of angle and time, pedantic and unnatural, as much as a decimal octave in music. Beware, then, of the Eiffel Tower, lest one day the French official, without warning to the sailor on the ocean, should flash out centesimal time.

The same thread and lump of chalk will carry us far in Dynamics; so that Dynamics can claim to be associated with Geometry in the boast of the *Principia*—"Ac gloriatur geometria (et dynamica) quod tam paucis principiis aliunde petitis tam multa præstet."

There is no need in this company to multiply instances, but here is one of Newtonian interest: Measure the length of thread which beats the second, with the lump of chalk at the end, or any other plummet. Every one here has a watch.

Look out in the *Nautical Almanack* the moon's mean sidereal period (27 days 8 hours) and the mean parallax (57 minutes); or better, perform the observations, as the easiest in astronomy.

Thence calculate the size of the earth.

The quadrant of the meridian works out to ten million times the thread, a little over 39 inches, about one metre, as decided in the metric system.

The piece of thread, or string, as the text-book will always call it, should not be used in statical experiment as a catenary, unless it is stretched taut, like a telegraph line; the thread is as inflexible as the steel wire. But the unpractical text-book tells us to use string for a catenary and not a chain, as not being so flexible or continuous, but composed of discontinuous links.

Well, here is the catenary made of a string, contrasted with a watch-chain of visible links, or with this other chain.

I recommend this chain to your attention for these experiments as flexible, practically continuous, and inexpensive.

Here is a few of the experiments in mechanics that can be shown with a length of the chain:

1. Catenary (illustrating the hyperbolic function).
2. Spherical catenary on a rounded surface (the third elliptic integral).
3. Skipping rope (elliptic function).
4. Conservation of angular momentum; drawing the ends apart; finishing with the vibration of a taut chain, rope, or cord.
5. Whirling a vertical chain (the Bessel function).
6. Let the chain drop, for the steady thrust of a stream of impacts, finishing up with the blow dealt by the part gathered up.

Forty years ago it was felt that the Cambridge School of Mathematics was too provincial, and outside examiners were called in—Maxwell, and Thomson, and Tait. A wonderful stimulus was the result on the enlargement of the outlook, in those the palmiest days.

But the effort died out with the disastrous division of the Tripos into two parts. The democracy got all it wanted in part one, and the second part died of inanition. A dreadful system has been copied everywhere, from Oxford, of three classes in alphabetical order; so that two-thirds of the candidates for Honours, however few, are dishonoured for life with a second and third class.

If order of merit is to go, abolish the class as well; so that a man who fails does not appear in the list. It is easy to award a mark of distinction in a subject.

But the examiner, prosaic and conscientious, considers himself tied down by his instructions, and is determined to have a third class in his list, however small, although the third class man may think he has stopped away.

This is called raising the standard, up to infinity. At the same time democracy is engaged in its occupation of lowering the standard for the mass, by cutting off one year of study, and that the most valuable. All achieving the same result in a different manner.

In the latest scheme, Part I of the Mathematical Tripos goes no further than the scholarship standard of the trained schoolboy. He will ask, and soon be allowed to take, this Part I at school, as well as the Little Go. The colleges will not object to being able to utilise Part I for the award of a scholarship and exhibition.

In the new second part a curious change is observable with the time of preparation democratised a year short.

The old problem demanding originality of thought is barred out, and the examiner finds he can satisfy the schedule

—simple, easy, elementary—

by mutilating generality, giving some special limited case.

There was a dreadful specimen last year in gyroscopic theory. The general theorem asserts that a top, spinning upright with angular velocity R, will be making

$$N=\frac{1}{2\pi}\sqrt{\left[\left(\frac{CR}{A}\right)^2-\frac{4gMh}{A}\right]}\text{ beats per second,}$$

and during the beat the top will make $\frac{R}{2\pi N}$ revolutions, and the apsidal angle will be $1+\frac{CR}{2\pi AN}$ of a right angle.

But the limitation of the examiner to infinite R was a leap into chaos, making the apsidal angle π; as the result is then applicable to any swift gyroscopic motion at any inclination, such as I can imitate here with this thread and plummet, and no definite conclusion can be drawn.

There was, however, a redeeming feature, where one examiner holds up a predecessor to scorn, by pointing out that the general case is as simple as the mutilated form given the year before; the problem being to discuss the oscillation of a pendulum body on any axle, fixed anywhere on a whirling arm.

But here the mischief of the schedule comes in. The examiner is barred from making the one step from his dynamical equation:

$$\left(\frac{d\phi}{dt}\right)^2 = A\cos 2\phi + B\sin 2\phi + a\cos\phi + b\sin\phi + h \tag{1}$$

to

$$\left(\frac{dz}{dt}\right)^2 = n^2(1, z)^4 = n^2 Z, \tag{2}$$

by the substitution $\tan\frac{1}{2}\phi = z$; so as to show that

$$nt = \int \frac{dz}{\sqrt{Z}} \tag{3}$$

is the general elliptic integral of the first kind.

No! because he would be wanting to traverse the water-tight bulkhead, set up by the schedule between the compartment of Dynamics and the Elliptic Function.

Posterity is jealous of anticipation of its verdict. But we must not let it blame us for not appreciating Maxwell at his true value, and in his lifetime. I have here three of his note-books, used in the preparation of his Lectures as a Professor at King's College, and for his Electricity and Magnetism. These will be as valuable to posterity as Newton's note-books would have been, used by him in the preparation of the Principia.

None of Newton's note-books have survived, and there should be four of these note-books of Maxwell. One of them is lost somewhere in King's College, and I invite the assistance of this learned company in a search for the recovery. It cannot have been destroyed, in a place of learning, as it calls itself. But when I write to ask for an organised search to be made, I get no answer, from the Principal of King's College downward.

I must not conclude and lose the opportunity of thanking the Mathematical Association for coming to our rescue in providing a committee to act for this country under the Central Organisation of Mathematical Instruction—I.-M.-U.-K. = Internationale-

Mathematische-Unterrichts-Kommission—which makes its final report at the Congress in Stockholm, 1916.

An application to our Board of Education for patronage met a very curt refusal—No funds! We know the Board has no money but what it cares to ask for. But why proclaim to the world that this poor country is beaten to a standstill, and content to crawl out a third-class power, when everywhere abroad we see triumphant progress.

Thanks, however, to the support of the Mathematical Association, and its assistance in organisation and finance, your representatives of England need not appear at Stockholm in *forma pauperis*.

The dramatic episode of Sundmann of Helsingfors should encourage the young mathematician as exhibiting the inexhaustible nature of our subject.

No sooner had Poincaré declared the Problem of Three Bodies insoluble than Sundmann showed how the divergency of the series required to hold for infinite eternity of Time, past and future, could be cured by a simple change of the variable.

But as Sundmann's memoir was published in the Finnish language of Helsingfors, the copy sent to Poincaré remained on his desk unread, and Sundmann's name is not mentioned in the Presidential Address at the Cambridge Congress of Mathematicians.

It was not till Mittag Leffler published a French version in the Acta Mathematica that the full importance dawned on the world, and Sundmann's merit was recognised by a prize of the Paris Academy.

So it is always open for the youngest of you to come to the front, where the veteran declares himself defeated.

DISCUSSION ON THE SYLLABUS.

THE following is a summary of the discussion which took place on the Syllabus.

Mr. A. Lodge: What I want to say is very short indeed: it is only to introduce a discussion in case any discussion is wanted on this Syllabus. There is one correction that ought to be made on the title page. The Syllabus was drawn up by the Public Schools Special Committee and approved "for publication" by the General Committee of the Mathematical Association. The approval by the General Committee merely amounts to sufficient approval of the Syllabus to request the Council to have it published. That ought to have been stated on the title page—that it was approved for publication.

There is very little I think I need say with regard to this Syllabus. Of course it speaks for itself. In Part I., which is for the Juniors in a Public School, I should like to draw attention to the proposal to introduce numerical trigonometry lower down in the School than has been customary in most Public Schools. The arguments for it have

been put fairly well in the Syllabus itself, and they are only carrying out in that suggestion the ideas given previously in a Report published in 1911, as you will see on page 2 of this Report. There is also another thing. With regard to the plane geometry, the Committee have felt that it is a great draw-back that illustrations from solids are not allowed. When I speak of "being allowed," I mean that in plane geometry examinations, problems directly connected with solids are not allowed to be set. The Committee felt, as probably we all agree, that plane geometry is really based on the study of certain aspects of solids, and it seems most desirable that in taking riders and problems in connection with plane geometry one should be allowed to take illustrations from actual solids. That is brought out in paragraph 7 of the Syllabus, Part I., on Geometry. It is very desirable that examining bodies should allow references to actual solids: boys would be much more interested, and girls, too, no doubt, if they could have illustrations in connection with actual solids instead of the plane geometry divorced from the solids to which it belongs.

Then, with regard to the alternative courses suggested in Part II., for Non-Specialists in Mathematics, *i.e.* senior boys in the school who are specialising for some other subjects. The suggestion is that they should be introduced to new ideas connected with *applications* of mathematics, or new ideas of interest in connection with actual *extensions* of their mathematics; but I think a strong feeling existed that it was better to put applications of mathematics, such as mechanics, in the first place. I think the Committee felt that this subject constitutes a very easy application for boys with no special mathematical ability, but of ordinary general ability, who wish to keep up some acquaintance with mathematical work, and to get new ideas in connection with it.

The next thing they take is Algebra. Very little is said about that in the Syllabus. I think the feeling was that the subject of greatest interest after Mechanics would be Calculus, and that for the reason again that in Calculus you are dealing with applications of mathematics to practical problems. That is all I wanted to call your attention to in the Syllabus, and I thought the important thing would be to see, if there is any discussion, whether people have any objections to it, or any suggestions to make for improving the suggestions that are made in the Syllabus itself.

Mr. A. W. Siddons: There are just two points I should like to refer to in connection with this Report. The great idea that the Committee is trying to keep before itself is to give freedom to teachers of different opinion to pursue their education as far as possible on their own lines, and not to be tied by examinations; and, secondly, this Report is really a preliminary Report to clear the way for further reports on the particular subjects. I think it is mentioned in this Report that reports on Arithmetic and Geometry will be produced later on—in fact, the Arithmetic Report is in a fairly advanced state, I believe.

Part I. lays down the course we feel everybody should go through, and then in Part II. we suggest various courses from which different teachers may choose one or two or more if time permits. I am very keen indeed that the solid which Professor Lodge refers to should be considered a compulsory part of the elementary course. I think it is a Report of the Board of Education which refers to the tyranny of paper having prevented people from seeing in three dimensions.

If any member has any remark to make, or any question to ask, Mr. Lodge will be very pleased to answer it.

Mr. C. S. Jackson: I should like, in the first place, to congratulate Professor Lodge on his recovery. I am afraid I am going to strike a discordant note, but I am bound to say I do not understand the principle

on which this Report is based. What view of mathematics is taken by the framers of the Report? My own position is this: about ten years ago I had a pretty firm belief that on the whole, what we may call the reformed methods of teaching mathematics were right, and I was a supporter of them. Time went on, and I began to entertain a doubt, and those doubts ended in a conviction, that many points must be reconsidered, and that in the main we must hark back. I should like most heartily to endorse what Professor Steggall said. It is not particularly interesting to learn to read and write, but you have to learn to read and write. If you are going to do any good with mathematics, you have to learn to go through the processes of elementary mathematics which correspond to reading and writing. I can give numerous instances confirming what Professor Steggall has said with regard to the complete breakdown in advanced work, which is simply due to knowing nothing about algebraic methods, and to inability to carry out the simplest computations in a decent sort of way. The fact is that the problem of elementary mathematics involves a very large number of different considerations. We have here one of the very greatest authorities in the country on the psychology of childhood, Dr. Nunn, and I am quite sure he will bear me out when I say that the problems of presentation are very imperfectly understood. For instance, there is a German proverb to this effect, that what Johnnie does not learn, John will never know, and I think it is profoundly important, and that it is the real justification for a good deal of the drudgery which we do not like to impose upon pupils, but which I am afraid we must impose upon them, if they are going to do any good afterwards. I believe the problems of mathematical education to be vastly more difficult, more numerous, and to require far more consideration than they have yet received. I also say that I believe it is profoundly important that this Association should not become identified with any one particular school. We stand for all persons interested in mathematics, and we must not allow the impression to get abroad that the membership of the Association means advocacy of one particular type of mathematical education.

Mr. C. Godfrey: Mr. Jackson has struck a somewhat pessimistic note; could he give us rather more definite information as to the particular points in the Report which are open to his criticism? My general impression is that the Report which is before the meeting cannot be called in any way an extreme Report. I believe that it was passed unanimously by the members of the Committee, and the members of the Committee were not, so far as I am aware, representative of any extreme school. They were elected by an open vote of the Association, and so may be considered to be fairly representative of the opinions of the members of the Association in general. I do not consider the Report extreme, and I should be glad if Mr. Jackson will give us some definite information as to the particular points which he wishes to have amended.

Mr. C. S. Jackson: I do not think I may speak twice, but I am quite prepared.

Mr. A. W. Siddons: I think it would be an advantage if you put your points now.

Mr. Fawdry: I think Mr. Jackson is allowing himself to be influenced too much by the type of boy he gets at Woolwich. It is my duty to send up some of these boys, and possibly I am responsible to some extent for their failures, but I think it is more likely due to the type of examination which they have to pass on entering Woolwich. I do not think it is due to the Syllabus which the Public Schools Committee have drawn up. Mr. Jackson complains that his boys know no Algebra.

Very likely the reason for that is that no Algebra is required in the examination which these boys have to pass, or practically none. I do not confess to being guilty of neglecting Algebra, and I suppose neither would the other army class-masters of Public Schools. Of course, we are largely influenced by the Syllabus of the examination which has to be passed, but we are also interested in the mathematical education of these boys, and although it is very difficult to find time for the consideration of Algebra, we find time for as much as we can, so that Algebra is not as much neglected as would appear from the character of the papers set for the examination.

Mr. A. Lodge: What we want to know is what part of the Report Mr. Jackson objects to—not the general tendency of modern mathematics, but what part of the Report shows that tendency to which he objects.

Mr. C. Jackson: I think I have stated my objection. I do not want to be led into quite minute discussion of detail. Of course, I might say I do approve A. and do not approve B. I do approve this, provided that certain other qualifications are borne in mind, but I do not think many of us would be better for such a statement. My complaint is that the problem has been dealt with in fragments instead of as a whole, and I am asked to discuss details ! I do agree that the entrance examination really combines in itself the basis of what I have been trying to argue. I said to Mr. Fletcher not long ago that, with regard to us in Woolwich, we had been living on the honesty of the schoolmasters who have taught mathematics to their pupils, and have refused to come down to the entrance examination. We cannot expect that to go on for ever, but that is what has gone on. I do not want to pass from one extreme to the other. The thing was wrong fifteen years ago and it is wrong now. If you could get half of what we had fifteen years ago and half of what we have now you would not be far wrong. My objection to the Report is that first of all I think no Report is of any value which does not lay down—perhaps I should not say that—the value of the Report is very much diminished unless it embodies the view which the people who frame it take of mathematics. The whole effect of this Report seems to be rather towards cutting down the position of the subject. I do not propose to discuss it detail by detail.

Mr. Andrade: I think one point of importance occurs here. It seems to me plain that an endeavour has been made to make the principles of mathematics, which in themselves require concentration and real thought on the part of the pupil, too easy, and to allow the pupil to pretend to himself that he has obtained a grasp of them by doing something very much simpler than he should do, and by doing it without thought. I think that is one thing that comes out clearly, and I agree that great harm is done by having too little put down and by having the prospectus of an examination made to appear far too simple, perhaps in many cases far simpler than it really is. I have had many cases of boys going in for examinations. They get hold of the Syllabus and they say, "We only require these things for the examination," and they do not exactly refuse, but they have no interest in anything beyond that. My experience of such boys has been that, once they begin to refuse to do any new work, or to be unwilling to do any new work, then term after term their knowledge of mathematics decreases. I have had that very markedly shown in a few cases with boys, such as I like to have, boys with strong views of their own as to the work they should do, and by diminishing the scope, mainly on account of these limitations on their work, and concentrating themselves on the more elementary parts of the subject, they seem to lose their mathematical knowledge instead of gaining more.

Mr. A. W. Siddons: Before I call upon Mr. Lodge to reply, there are one or two points I should like to make. The last speaker spoke about boys working without thought. I have been a schoolmaster for fourteen years now, and I should say that the real essential difference between mathematical teaching of to-day and mathematical teaching of fourteen years ago, is that to-day there is much more thought put into the work and much less turning the handle than there was fourteen years ago. I shudder to think of the mechanical way in which I myself taught when I began teaching, partly through ignorance, and partly because it was the process which I found in vogue; and I should say that to-day we are really aiming at making people think; we are not making them do so much as they used to do, but we are making them think the more. Another point which occurred to me when Professor Steggall was speaking was that there is a tendency to put off manipulation. Now, I do not consider that manipulation is part of the necessary equipment of the average boy. It is a most essential piece of the equipment of the mathematical specialist, and I hold that every Woolwich boy is a mathematical specialist. Those boys should have a considerable power of manipulation. The question is, Where are we going to give them that power; and I hope that this Association in some of these Reports will make a point of answering the question. Should it be given in what you may call the college course, or the school course? Should it be given in the early part of the course, or should it be given later? I think the Report has already implied that it should not come into the general course for all pupils, including non-specialists. But there is a question which has to be faced—the question of manipulation. I feel very strongly that manipulation is not an essential tool for the use of the ordinary non-specialist. He should not be given a power of using a particular muscle; he should be given ideas which should exercise his mind rather than a muscle.

The President: Now I will call upon Mr. Lodge to reply.

Mr. A. Lodge: I think, as a matter of fact, I have got nothing to reply to, because there has been no comment made on the Report itself. With regard to the actual comments made, I think Mr. Jackson's feeling was that, although he has not definitely spoken of details in the Report, he has been feeling for the last year or two that some of the simplifications that have been suggested in connection with Part I. of the Algebraic work have been suggested because there was a fear of the pupils being taught formal mechanical work rather than going on to new ideas. I agree with Mr. Siddons that the feeling in the Report has been: "Do not spend quite so much time over mechanical skill in Algebraic manipulation, but instead of that get on to new ideas and introduce, for example, Trigonometry rather than going on to the harder part of Algebra." There is nothing in the actual Report itself saying that mechanical skill is not to be insisted on, and I suppose it is for that reason that Mr. Jackson could not actually pin us down to some particular misstatement that, as a matter of fact, we have not made.

Then, with regard to what Mr. Fawdry said as to the fault of the examination at Woolwich. Certainly it is a fact that the questions for the most part appear to aim more at testing the ability of students to think, rather than at testing their mechanical skill. The problems do require thought, but the actual mathematical methods which have to be employed may be for the most part of the rudimentary type. The mathematical skill asked for in the entrance examination is not very great, and, as Mr. Fawdry says, the teachers take care that there shall be a good deal more mechanical skill taught than would have been required merely for the entrance examination, because they have to think of what the students will do when they get to Woolwich.

I hope Mr. Jackson will agree that many teachers have managed to send students who possess mathematical skill, in spite of the entrance examination.

Then, with regard to the Syllabus and the question of students going beyond the Syllabus. I must say, again referring to the Woolwich examination, that the Syllabus for that examination goes quite far enough: it is only that the actual papers do not test the student's knowledge up to the limits indicated by the Syllabus. The Syllabus is very good, and much better than it was seven, eight or nine years ago. Then we did have cases where students objected to doing certain work, because it was not in the Syllabus, although we knew that when they got to Woolwich the knowledge would have been most useful to them. That was in the early days of the Woolwich Syllabus, before the extension. At present we have no trouble of that type.

With regard to Part II. of the present Public School Syllabus, which is for non-specialists, that has certainly been framed on the lines mentioned by Mr. Siddons, on the ground that non-mathematicians are more interested in getting new ideas than they are in getting mere mechanical skill, and on the ground that it is probably more important for a man whose after life will be occupied with philosophy or things not mathematical that, to widen his general outlook on life, he should have absorbed ideas connected with mathematical science and philosophy rather than have had his school time spent in acquiring greater skill in mere manipulation.

A Member: May I ask if any of the changes in the Syllabuses include the introduction of the elementary ideas of the Calculus ?

Another Member: I should like to ask whether the Committee is simply going to attend to English examinations. We are still a United Kingdom, and there are examinations in Scotland and Ireland which might be equally worthy of the attention of the Committee. Is there any prospect of the facilities of the Oxford and Cambridge Locals being extended to those examinations ?

Mr. A. W. Siddons: I should like to say one word about the policy which the Committee has had before it. It has tried so far as possible to go for freedom; I mentioned that this morning; we have tried to urge these bodies that they should set alternative papers, so as to give to people, who are teaching on different lines, opportunities of being examined on the lines they are teaching on.

The second question was with reference to limiting ourselves to English examinations. I am afraid so far we have done so. One of our difficulties has been that some of our members have been at considerable distances, and naturally it is difficult to come to meetings; but I think we must try to make some attempt to get into touch with other examinations, and if people at a distance will write to the Secretary and make suggestions as to the way to proceed, I am sure it will be of value to the next Committee. The direct method of attack is not always the most suitable, as I think the Committee has found to some extent. It is always necessary to enquire privately of somebody who knows what is the best way to do it. One body says: "Oh, you just send us a private letter and get us to ask you for suggestions"; and another body says: "Fire them off at us at once." I am afraid at London University we were not successful. We took a lot of trouble to find out what was the proper method of attack, and nobody seemed to know, and the method we were ultimately advised to take has not proved successful.

Then, with regard to the question of the examining body not requiring proof of certain propositions in elementary Geometry. We were asked if there was any chance of getting that extended to the Oxford and

Cambridge Joint Board Examinations. I think the Committee will naturally press that upon the Joint Board, and I should think they would consider it very favourably. It is a little unfortunate, I think, that, although some examining bodies have recognised the principle, they have not chosen the particular propositions that we should like them to have chosen—they have not chosen the propositions, or at least not all the propositions, that the Committee recommended, and they have put on the list some that we did not ask for; but still it is their acknowledgment of the principle which may be of great use in the future, and we hope that they will come to recognise which are the ones which we want to be on that list.

If there are no other proposals about this Report, then I shall put it to the meeting that the Report be approved. (Carried.)

THE VARIOUS USES OF GRAPHS.

It cannot be said too often that each portion of a mathematical curriculum must be chosen with some definite purposes in mind, and that these must be the controlling influence in the scheme of development. Such purposes may be roughly divided into two classes: the first includes all desires to illustrate or elucidate general ideas and principles, and the second includes all desires to relate one part of the subject to another, or to some other domain of thought. It is not here implied that possible fulfilment of some such purposes enforces the inclusion of any particular branch of mathematics; its difficulty may be too great for the students in question, and in any case only a limited amount can be included in a given time-table. But it is very definitely intended that no branch shall be included merely because it happens to be sufficiently easy or to tickle the fancy. There is ample real and human interest to be found without sacrifice of purpose; and while undue difficulty is to be deprecated, it must always be remembered that definite achievement resulting from continued effort is of the very essence of the learning of mathematics.

The object of this paper is to consider the purposes which may be served by the inclusion of graphs in the curriculum, and in particular to show that they fall into each of our two classes; that is, some of them are concerned with the presentation of general ideas and principles, while others are concerned with the interrelation of various subjects. The uses of graphs extend far beyond the approximate solution of equations, finding of maxima and minima, and the like. Properly developed, they provide the best introduction to the ideas of functionality and elementary analysis; and further, they show the relation of algebra to these ideas in a manner peculiarly clear and convincing.

When one looks at a graph of any particular phenomenon for the first time, what is the first consideration which enters the mind? Despite the laborious efforts of examiners and writers of books, a man who understands what a graph is does not first look at the details of axes and scales. He places it right way up—I admit that this involves a momentary consideration of the directions of the axes—and examines its general shape, thus gaining at one glance an outline of the whole sequence of events which is being considered. The graph is not the only way of gaining this information, any more than the formula $\frac{4}{3}\pi r^3$ is the only way of learning how to find the volume of a sphere; but it is the most vivid and enlightening method, just as the formula is more vivid and enlightening than an involved verbal statement of arithmetical processes. We may say of a certain plant that it grew slowly at first, then more rapidly, then more slowly, and finally reached a maximum height, but the appropriate curve gives this information more rapidly and effectively.

Hence the first use of a graph is to give information by its general shape, and so the first aim in teaching should be to enable the pupils to say what various shapes indicate about given familiar phenomena, and to sketch shapes corresponding to events whose sequence is described. For example, they should be able to interpret a graph as illustrating the changes in weight of some individual, and to sketch an illustrative graph when they are told that a man had an accident necessitating an immediate amputation, following which he had a long illness, and recovered completely. No axes should be marked, and no numerical idea need be involved.

Although it is not of great difficulty, this discussion of shape is of the utmost importance, and deserves full and careful treatment. Apart from a vital application in the drawing of graphs, to be mentioned later, it forms the true origin for the concept of a function, for the sequence of changes is exhibited as one whole and appears as a distinct entity, namely, a function. Of course, in the illustrations which are best suited for young pupils, the function idea is not exhibited with great precision; a boy's height is a function of a good many other variables besides his age. But the essentials are clearly set forth, and that is sufficient. There is a well-known remark of Klein, to the effect that it is unwise, and even impossible, to be entirely rigorous in teaching the young. Like most sayings of equal truth and authority, this can be quoted to justify all manner of processes which are indefensible under any circumstances; but we are certainly on safe ground in quoting it to support the development of the function concept from familiar phenomena, without any meticulous examination of detail in the early stages.

We may regard graphical expression as a language, just as we regard algebraic expression as a language. An individual graph is a sentence in that language, just as an individual formula is a sentence in algebraic language, and this graphical sentence is a descriptive account of the properties of some function. Just as we now begin to teach a foreign language by developing some power of expression before examining the niceties of accidence and syntax, so should we teach graphical language by developing an understanding of the shapes of graphs before examining the numerical niceties of axes, scales, and the like. This understanding of shape implies an ability to perceive increase and decrease, maxima and minima, discontinuities and the like, and to distinguish between greater and lesser rates of change by regarding the steepness of the curve. No numerical estimates are involved; the work is essentially qualitative, not quantitative. To lead up to the qualitative idea that steepness corresponds to rate of growth by involved calculations of gradients is very much akin to proving that two sides of a triangle are greater than the third side by the assumption of a result which is no more obvious than this. It is from the spontaneous connection of steepness with rate that the measurement of rate by means of gradient should be developed.

Here I must digress for a moment to refer to the standard directions of the axes. Judging from the experience of most examiners, there appear to be many teachers who regard deviation from the accepted practice in the same light as the use of varying letters in the figures for geometrical propositions. Now this idea is wholly erroneous. If anyone chooses to state his intention, he can perfectly well write the number consisting of 3 tens and 5 units as 53, or $\frac{3}{5}$, or $\frac{5}{3}$, and so for other numbers; and if his statement is remembered, his calculations will be intelligible. But the essence of arithmetical convenience is that we all choose the same plan; and so also with the essence of graphical convenience. There is no essential virtue in the method actually adopted in either case: to think that there is, is to make the mistake of the old lady who praised Noah for giving all the animals their right names. But there is a very essential virtue in the universal use of the same method, for it is the only way of enabling one individual to gain information from the work of another, or to review his own past performances, with any degree of facility.

All qualitative information having been obtained from the shape of a graph, the next step is to examine its quantitative properties in order to obtain any numerical information which may be desired. Without touching on consideration of gradients or areas, which are beyond the scope of quite elementary work such as is under discussion, the graph can be used to obtain the following information concerning the magnitude considered:

When it has a given value.

When it is greater or less than a given value.

When it lies between two given values.

The range of values included during the period considered.

When it is greatest and least, and the corresponding values.

When it changes from increasing to decreasing, and *vice versa*.

When it is growing more and more rapidly, and when more and more slowly.

The time taken to increase or decrease by a specified amount.

Unless all this information can be obtained from a graph, its quantitative uses remain unrealised, and most of its stimulus to thought and investigation is lost. This is largely the case at present. The excessive and absurd concentration on the one matter of finding when the magnitude has a given value has obscured all other considerations, many of them far more important, and most students have little idea of the range of expression of this very simple language.

In what has already been said there is, of course, no allusion to graphs of algebraic expressions; we have been only concerned with curves illustrating definite and familiar phenomena. And further, only the uses of the curves have been considered; the methods of construction remain to be explained.

The problem of drawing a graph, however it may arise, is always presented in the same form. Information is given concerning a number of discrete points, and the curve is then drawn as accurately as may be possible. If the information arises from statistics concerning some phenomenon, it may be that no more can be obtained; but if it is obtained from an algebraic expression its amount is unlimited, for we can compute as many values of this expression as may be desired.

The various points having been plotted, it remains to draw the curve, and it is here that the preliminary consideration of shape assumes vital importance. Its bearing on the completion of the curve is best illustrated by an example.

Suppose that the depth of water at the end of a pier is observed at noon on each day for a fortnight, the first being taken at low tide, and the results indicated on a graph. A set of 14 points will be obtained which suggest a curve in the shape of an arch. The unthinking student will promptly complete this curve, and imagine that he has a graph showing the depth of water at any time during the fortnight; but in this he is entirely wrong. The tide rises and falls twice in a period of rather more than 24 hours, and the true graph is a series of undulations which pass through the marked points in approximately this manner.

FIG. 1.

Here we see fully exposed the fatuous nature of the customary direction to "plot the points and sketch a smooth curve through them." If followed, it leads to hopeless error. The only scientific method is to sketch the curve through the points, *having regard to what is known concerning its general shape.*

Of course, in many cases it is impossible to predict any general shape, and then this direction must be followed as a counsel of despair, provided that the points do suggest any definite curve. But this should be done with an admission of large assumption, and attention should at once be directed to obtaining further information concerning intermediate points. At the same time, information of a general character is often a considerable guide to shape. If four census returns of the population of a town give four points situated as in Fig. 2, a certain curve is suggested; but if it is also known that there was a sudden influx of population between the second and third return, the curve must be drawn as in Fig. 3.

FIG. 2.

FIG. 3.

It is mental exercises of this kind which produce a man who can think mathematically concerning phenomena, rather than one who can merely perform, without any understanding, certain comparatively trivial devices of the engineer. The automatic drawing of smooth curves for little or no obvious purpose except to find some special value is very nearly as deadening as the most formal type of algebraic exercise; and it is far more dangerous, for it induces a feeling of competence and reality which has little justification.

It will now be seen that graph drawing of this type does really induce the idea of functionality; the general shape of the curve indicates the general nature of the function, and its numerical details fix with more or less precision the numerical values of the function for different values of the variable. It is in this connection that the term *function* and the functional notation should be introduced, no unnecessary formality of definition being required. The pupil should be able to say whether one number is or is not a function of another; for example, that the number of hairs in a man's head is a function of his age, but is not a function of the number of hairs in his wife's muff. And further, when the details of a particular function are specified by means of a graph, easy equations should be solved, for example,

$$f(x)+3=f(5), \quad f(10)=xf(5), \quad f(10)=3f(x).$$

Such work gives a reasonable amount of precision and completeness to ideas which have now been formed.

Of the construction of graphs of algebraic expressions little need be said here. But it may be suggested, that given a familiarity with graphs and their shapes such as has been described, some convincing proof of the shapes of simple algebraic graphs is not so hard as might be imagined. In the case of expression of the first degree, for example, $3x-4$, we have

$$\{3(x+1)-4\}-\{3x-4\}=3,$$

and hence the rise in the graph for a unit step to the right is the same *wherever that step is taken*. Few will then doubt that a straight line is the only possibility.

For a quadratic function, say x^2-3x+5, we proceed similarly. The calculation

$$\{(x+1)^2-3(x+1)+5\}-\{x^2-3x+5\}=2x-2=2(x-1)$$

shows that the rise for a unit step to the right increases steadily as the step

commences further and further to the right of $x=1$; the curve must therefore ascend more and more rapidly, without any undulations.

I touch on the graphs of algebraic expressions thus lightly, because the real crux of the use of graphs in mathematics lies in the relation of these expressions and their graphs to the functional ideas already evolved, and the discussion of this relation is the real purport of this paper.

It cannot be said too often, or with too much emphasis, that the concept of a function is something quite different from, and much larger than, the concept of its algebraic expression. Important as is this consideration to the pure mathematician, it is of even more importance to the teacher of elementary mathematics in schools, for the functional relations which are the concern of most adults—I do not mean vocational concern—are seldom such as can be expressed in algebraic terms. Population, mortality, expectation of life, the value of money—none of these can be summed up in a formula, and it is with such types that the ordinary adult is really concerned.

And yet it appears to be often thought—and almost invariably taught—that *function* and *algebraic expression* are equivalent terms. In almost every text-book we find $f(x)$ merely standing for some algebraic type, the general idea of dependence being entirely ignored. The pupil may well think, as he often does think, that function is another word for formula or expression, that the whole matter is something which is dragged into algebra without apparent purpose, and rather a nuisance at that.

To enforce the distinction between function and formula, let me give one illustration. The weight of a body is a definite function of its distance from the centre of the earth; but this one function is represented by different formulae according as the point is above or below the surface of the earth. If above, the formulae is A/x^2, and if below, Bx, x being the distance from the centre of the earth.

If we say that there are two functions giving the weight, one for points inside the earth and the other for points outside, we are merely adding another word to the two, expression and formula, which already exist for one entity. The law of dependence defined by variation of weight with distance from centre is something quite different from either. It is a law, not the expression of a law; and this it is which is denoted by the term function.

The essence of algebraic expressions is that they form the ideal method of functional statement. The graph forms the first method; its want of precision is compensated by the vivid and terse manner in which it conveys the idea of dependence, and the nature of any particular dependence. But these having been acquired, no one doubts that the algebraic expression of dependence is the final type, alike in exactness, convenience, and extent. A formula is more precise and handy than any graph, and it reaches much farther.

This use of algebra is easily exemplified by simple illustrations, such as the dependence of the area of a circle on its radius. The area should have already been considered as a function of the radius, and the general nature of this function is therefore familiar; for example, since successive increases of the radius by one inch give larger and larger increases in the area, the graph is a curve which becomes steeper and steeper as we pass to the right; it can therefore be sketched from a few special values obtained by measurement of definite circles. But as soon as the formula for the area is obtained, any ordinary pupil can see that it provides the ideal expression of the dependence of area on radius. It is as much superior to the graph as the graph is itself superior to a table of values for specified radii.

Illustrations in which different formulae are necessary for different values of the variable, as in the case already mentioned of the weight of a body, should not be omitted. They emphasise the distinction between a function and its algebraic expression, a distinction which is essential to any rational comprehension of a function as an entity in itself.

The student is now in possession of two means of expressing functional dependence, graphical and algebraic. The first is illuminating as regards general nature, and fairly accurate when sufficient data are known; the second is ideal in its precision and compactness. It may be described as the coping stone of the first. But only certain very special functions are capable of this ideal expression; the functions of everyday life cannot be stated precisely by formulae. Nevertheless, formulae can be constructed to represent any function with an accuracy which is only limited by our detailed knowledge of individual values of the function, so that there is no such severance between functions which have an exact algebraic expression and those which have not, as might be supposed.

The method of adjusting formulae to given statistics is of course well known, and it is only necessary to mention one case. Suppose that a certain

FIG. 4.

graph is known to have the shape in the figure. Such shapes are given by the expressions

$$y=\frac{bn}{(m+n)t^m}x^n, \quad 0<x<t,$$

$$y=b-\frac{bm}{m+n}x^{-n}, \quad x>t,$$

$x=t$ being the point of inflexion, that is, in words suitable for pupils, the point where the curve first becomes less steep. The constants m, n, t, b can be adjusted to suit the observations in question by solving very easy equations, and it will be found in practice that the formulae give satisfactory results if the order of accuracy of the given data is not very high.

All other simple types can be dealt with on similar lines. The work provides excellent drill in easy algebra of a useful type, and the choice of appropriate formulae enforces valuable forms of thought.

One particular case deserves further mention, in that on no account should it be omitted. The student should draw in one figure the graphs of $\sin x$, x, $x-\frac{x^3}{6}$, $x-\frac{x^3}{6}+\frac{x^5}{120}$, ... and realise how each successive term in the formulae draws the graph round into another wave, the process being obviously capable of indefinite extension; and similar work should be done for $\cos x$. The results can be summed up thus:

The formula x gives the function $\sin x$ to 2 decimal places from $x=0$ to $x=$

" $x-\frac{x^3}{6}$ " " " " "

and so on. There are few processes which give more insight into the interrelation of functions and formulae than this, and the idea of obtaining as many waves as may be desired, to any accuracy that may be desired, is also most valuable.

It may be asked whether a student should be given such formulae without any clue to their origin. In this case there is ample justification, for their use and meaning are clear, and the statement that they are obtained by the

aid of other parts of mathematics, which he may hope to learn himself, gives an outlook for the future. Many boys, and not only the most able, are impressed by such a sense of vista.

It will be seen that I have suggested that the whole function concept should be evolved from general, and not algebraic, considerations, algebraic expression being regarded as the ideal method of expressing functional relations, not as the normal way of conceiving a function. The same applies to the processes of the Calculus. These should be developed, and their notation explained, on similar lines, the algebraic processes appearing as the ideal form of statement for ideas already familiar. It is one more illustration of the fact that the proper methods of developing pure mathematics in abstract form, if only they are expressed through familiar entities and ideas, are also the proper methods of developing mathematical ideas in education.

Many advances have been made in teaching practice during the last fifteen years, and mathematical education has, to some extent, benefited in consequence. But potential gain in all branches has been in large part sacrificed by lamentable want of any considered theory, and not least so in the treatment of elementary analysis. We are often told that the idea of functionality and the processes of the Calculus can be amply illustrated by study of very simple algebraic expressions such as x^n. Now I will venture to say, and I have endeavoured to show it in this paper, that this view is based on a most vicious misconception of the true meanings of functional dependence and algebraic expression; that it is mistaken educationally is well known to many who have experience of its results. The genesis of the error is easily perceived. The old method of development from the abstract, typified for example in the text-books of Todhunter, has not really been abandoned. It is still seen in this attempt to develop functional ideas on one simple abstract function, namely, x^n. These ideas should be evolved from concrete illustrations, and then should find their abstract expression in algebraic formulae. To commence with formulae, and illustrate them by concrete examples, is still to put the cart before the horse, and this is what we are now doing.

Moreover, it is at least doubtful whether a boy gains any real power of thinking about rates and total effects by learning to differentiate and integrate x^2 and apply the processes to simple problems. To learn to think about such processes he must evolve them from familiar concrete illustrations of all kinds; in this way only can he become familiar with them. And when they are familiar tools, which can be applied to any matter in hand with some confidence, their algebraic expression may be developed as the apex of the whole structure. But if this is done, it must be done with some thoroughness. It is mockery to say that a boy can realise the practical power and abstract beauty of algebraic expression as applied to functional processes by one very simple case. We might as well say that he can appreciate the structure and beauty of Latin by learning to ask for his bare needs in that language. Familiarity with functional ideas must be acquired by ample exercise in the study of definite functions, which are defined by familiar concrete relations; and familiarity with their algebraic expression must be acquired by ample exercise in that expression; no other course is possible.

I have spoken somewhat strongly because I feel, as more and more teachers are coming to feel, that we are now drifting on to the rocks for want of any definite policy. In the old days, some boys did learn to do something, and no one knows how largely the number might have been increased by more efficient teaching on similar lines. But that efficiency has been destroyed —as was well said recently in the *Times*, one defect in modern systems of education is the substitution of the teacher's skill for the pupil's labour in self-training—and nothing stands in its place. What is needed is, first, a power of mathematical thought about common things developed in all directions as far as possible, and secondly, some real knowledge of and control over the abstract methods employed by mathematicians to express that thought;

and the two can only be acquired in this order. At present, we reverse the order, reduce the second to mere triviality, and so ruin the first, and the result is well known to many. Boys cannot do the simplest piece of algebra with any efficiency, and the gain in intelligent thought is quite insufficient to form any compensation. It rests with the mass of teachers to make up their minds what mathematics really is, and why and how it should be taught; and the need is urgent.

G. St. L. Carson.

DISCUSSION.

Mr. Daniell: I should like to ask Mr. Carson if, in using the term "functional dependence," he had in his mind any idea of causality? It seems to me that the graph expresses correlation only.

In general, I would support what Mr. Carson has said, particularly about the study of the aspect of the graph. It often falls to my lot to read examination answers dealing with the subject of graphs. It does not much matter whether candidates have been told to give the form and name of the graph that they have drawn; they volunteer the information, and very cheering it is sometimes. On the last occasion on which I asked some candidates to plot a graph from measurements, they informed me that the result was a straight line, a circle, an ellipse, and a hyperbola, the curve being actually a roulette of sorts. This description was not a function of the data provided by the question, but of the particular lessons they had recently had. It is well known that quite eminent discoverers have made the mistake of smoothing their results too much in order to get a curve, a mistake which was so powerfully illustrated in the example given by Mr. Carson.

Mr. A. Lodge: In continuation of Mr. Daniell's remarks, I should like to ask Mr. Carson whether he would not feel that the idea of function is the idea of causality, but when you get to a formula you only have correlation. In thinking of the idea of the function you are thinking of causality. With regard to Mr. Carson's first illustration, in which he first of all gave two things which apparently had no connection at all—the number of hairs on a man's head and the number of hairs in his wife's muff—certainly he suggested it by means of a formula—but the causality was the first thing which came into our minds when he suggested that the numbers were in inverse ratio. I have often felt that when you use the word function in mathematics you are really thinking of some formula or another. When you say there are certain functions, you usually mean certain mathematical expressions or formulæ, but Mr. Carson's statement that the function is so different from the formula, rather suggests that with the idea of function you are thinking of causality, and in the case of a formula you are only thinking of correlation.

A Member: May I ask quite a different question on a point of practical detail? I did not quite understand how far you could explain the meaning and shape of a curve until you had explained the nature of plotting it—the different methods by which the graph is drawn, so as to give any indication of the increase in the function. It seems to me that we must start off with our axes and explanation of the method properly before we can explain at all why the alteration in the curve indicates the increase.

Mr. Andrade: I should like to suggest one other curve that could be given to suggest very clearly the shape that is wanted rather than anything else. I very often give my students to draw a hunger-time curve—the feeling of hunger at different times—and I find that rather interesting, because it leads sometimes to a discussion as to whether

during your meal your hunger diminishes at a greater rate at the beginning of the meal or at the end.

Mr. Carson has warned us that it is not safe to join up a few plotted points by a smooth curve unless we know from other sources that the graph should be a smooth one. What would he do if a few points were given and no other information regarding the function could be obtained ? Would he join up the points by a number of straight lines, as doctors do in temperature charts ?

Mr. Carson: Mr. Daniell and Mr. Lodge both raised the same point, that is, whether I regard functional dependence as based on causality or correlation. The answer depends on the individual who is thinking of functional dependence. Causality is a concept which becomes familiar to human beings quite early in life. It can be appreciated by a boy or girl, in simple cases at any rate, and therefore the original idea of functionality should be developed mainly from notions of causality. All concrete illustrations, such as I have mentioned, involve causality rather than correlation. I am not an expert in these matters, but I should say if a formula is regarded merely as a correlation, nevertheless the proof of the formula may come very near to involving causality. I do not think the two can be separated in elementary teaching. I am very suspicious of any attempt to say in teaching " Here we are going to deal with causality and there with correlation," for the one thing to do is to build in every possible way on the pupil's own ideas, and these involve each factor almost indistinguishably. I believe, however, that in early years we shall in practice rely on impressions of causality rather than on correlation. Mr. Lodge, I noticed, said that we ourselves instead of thinking of " function " usually think of " formula." I agree that very often we mathematicians, in attempting to think of functions, do think of formulæ, but I do not think, we are wise, and certainly it has been shown in recent investigations that the idea of functionality and its development is most simply evolved without reference to formulæ of any kind. All the modern courses of analysis proceed thus, and what I suggest is that the same procedure should be followed in elementary teaching, concrete illustrations being used throughout.

I was also asked how the meaning of the shape of a graph can be explained before any points have been plotted with axes and scales. What is in my mind is that the graph idea should be evolved from consideration of continuous automatic recorders of all kinds ; barometers, an automatic recorder of rainfall, where a jar is exposed to the rainfall, and its weight depresses a pen which marks a revolving drum ; an automatic recorder of the growth of a plant, made by fixing a pen to the stem ; and so on. By such means the idea of a graph can be developed, starting from a complete curve instead of a series of plotted points. I find from experience that this can be done with pupils younger than I should have thought possible.

The final question was with regard to the matter of joining up points. What I intended to say is, that if we do not know anything about the general shape of the curve, and if the points plotted do suggest some curve, then as a counsel of despair we may sketch that curve, and at once try to ascertain, by obtaining more data, whether the curve is right or wrong. But if the points suggest no curve, or perhaps even if they do suggest one rather vaguely, the right course is to leave them as plotted, a series of discrete points. No one lie is much better than another, but what I will call the straight-line lie is perhaps the most despicable of all, because it is the most easily told and is farthest from the truth.

Sir G. Greenhill: Mr. Carson spoke of the difficulty of constructing formulæ to suit the operations of every-day life, which reminds me of

a formula. I am afraid you will think it trivial and of no utility for teaching in a school. The formula is $Y=\frac{X}{2}-7$. X is the man's age, and Y the age of the girl of his choice. If you substitute in that formula, and Mr. Carson draws the graph, in the moderate range of human life, it works out in a good manner. But my experience is that when I have given the formula in the inverse manner of interchange of the variables, by saying to her, "The girl of my choice ought to be 40; how old am I?" she will not reply. She suspects something behind it.

We thank Mr. Carson for his interesting paper, and now Dr. Shaw, who is present, will read us his paper on "Principia Atmospherica." *

ON THE SETTING OUT OF CERTAIN EASY CUBICS.

If we wish to develop a race of geometers, I think we should begin by encouraging the habit of speculation. This we do nowadays by using squared paper for plotting graphs. Here, however, we place ourselves very much in the hands of the printer: and although ideas of geometrical shape are enlarged and systematised, the right angle plays too great a part in whatever knowledge of curves is acquired; indeed a knowledge of curves is not the professed object in plotting graphs so much as a visualised representation of corresponding changes in related physical quantities.

In teaching elementary geometry there lingers an aversion from the use of dividers in pricking out equal distances: for the transference of distances is not recognised in the postulates of Euclid, and Euclid's influence remains.

Later on a study of the conics, carried on for a prolonged period, and the application of the theorems of Pascal and Brianchon, must exercise their fascination. Ideas about infinity become clarified: the use of parallels is seen to be necessary, to exercise the right accorded by Euclid's first postulate to join any given point to any other, even if one be at infinity. But the mind inevitably tends to become obsessed with the second-degree idea. Mere straight-edge methods are found to be so potent in the construction of conics that metrical properties are neglected. Primary notions about equal distances interfere with the whole scheme of a projective geometry confined to the conics; and the bisection of a given finite straight line becomes a search for the harmonic conjugate of the point at infinity in its direction, with regard to its two ends.

By being encouraged in the use of parallels and dividers the young mind may be led to a quantity of information. Experimental geometry leads to speculation, speculation to information, information to knowledge.

Among the earliest notions that can be developed in this way is that of symmetry. We recognise the value of encouraging an intuitive sense of symmetry, nowadays, one is glad to say, in quite elementary courses of geometry. But I do not know that boys and girls are ever required to pick out the points that are symmetric correspondents, with regard to a straight line, of other points. If they are it is probably always rectangular symmetry; *i.e.* the parallels are always slid perpendicular to the axis of symmetry.

I think we might usefully recognise a geometrical property of figures, which I will call oblique symmetry: got by sliding the parallels at a constant angle to the axis of symmetry. In this way a scalene triangle can be seen to be obliquely symmetrical about each of its medians. A parallelogram has four axes of oblique symmetry. The "regular" figures have none.

With the use of dividers only, the conic hyperbola can, I think, be introduced at a much earlier stage of elementary geometry, if the speculative idea is to be encouraged. I would begin in this way:

* The paper is not printed in this number of the *Gazette*.

Take any pair of fixed straight lines (Fig. 1) and select a point A at random in their plane. Draw any straight line ZAZ, and prick off $ZB=AZ$. Do this many times and watch the locus of B. This will suggest a great deal of useful discussion. At first a single branch of the hyperbola will be found, and it will be seen to go to infinity in the directions of the fixed lines.

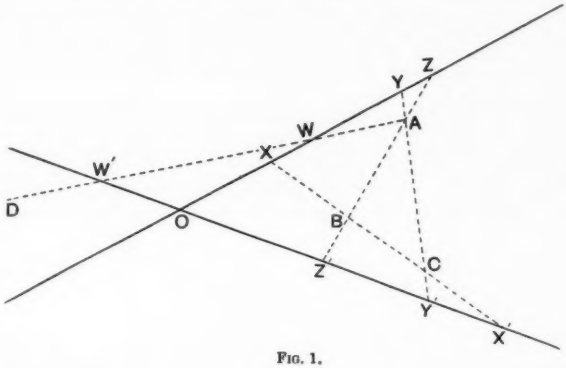

FIG. 1.

So far it appears to be a "second-degree" curve, because every line through A cuts the curve in only two places. Do any lines cut it in three places? If we draw YAY', and prick off $Y'C=AY$, shall we, by joining BC to cut the lines in X, X', find that $X'C=BX$?

Careful drawing and measurement may illustrate the truth of this. If we have got as far as Menelaus' Theorem we can then say:

$$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{BZ}=1=\frac{CX'}{BX'}\cdot\frac{AY'}{CY'}\cdot\frac{BZ'}{AZ'}.$$

But
$$\frac{CY}{AY}=\frac{AY'}{CY'},\quad\text{and}\quad\frac{AZ}{BZ}=\frac{BZ'}{AZ'};$$

therefore
$$\frac{BX}{CX}=\frac{CX'}{BX'},\quad\text{and } X'C \textit{ is} \text{ equal to } BX.$$

This shows that the curve we have found (as far as real points are concerned) is truly of the second degree.

To go further, instead of saying $ZB=AZ$, $Y'C=AY$, $X'C=BX$,—which are all perfectly true,—we will say $AZ+BZ'=0$, $AY+CY'=0$, $BX+CX'=0$, and we can extend the curve. Draw AW, as in the figure; produce it to meet the other line in W', and take D so that $AW+DW'=0$.

In this way we shall arrive at the whole idea of the conic hyperbola. It will soon be manifest that it has two axes of rectangular symmetry, the angle-bisectors of the fixed straight lines; and any number of axes of oblique symmetry through their intersection, O, each bisecting a series of parallel chords. Without any knowledge of the focus-directrix properties of conics we get at all the fundamental asymptote properties of the hyperbola. Valuable notions on tangency come in. We see how by a very simple geometrical construction to draw the tangent at A, because its intercept between the asymptotes is bisected at the point of contact. We also see that it has a "centre," O, all chords through which are bisected there. We can soon, I think, see that a diameter through O parallel to the tangent at

A bisects all chords parallel to OA, and thus acquire the idea of conjugate diameters.

Then, when we have drawn the curve as far as we can, we can talk about its continuity, how it disappears to infinity in one direction along an asymptote, and reappears from infinity in the opposite direction *on the other side* of the asymptote.

And after a time, long before exhausting the properties of conics, we shall be ready for speculation about third-degree curves. How would it do if we were to take *three* fixed straight lines, and get a series of trios of related collinear points P, Q, R, such that if the transversal PQR cuts the three given straight lines in X, Y, Z, $PX+QY+RZ=0$?

It is obvious that a single given point P would not determine Q and R: and that two given points P, Q would only determine one other point R.

But if we are given three points P, Q, R (Fig. 2) not in a straight line, we can find three other points S, T, U in PQ, QR, RP respectively to satisfy the conditions. Then SR, TP, UQ, ST, SU give us five other points, and we can proceed to set out any number of points on the curve. The points can be

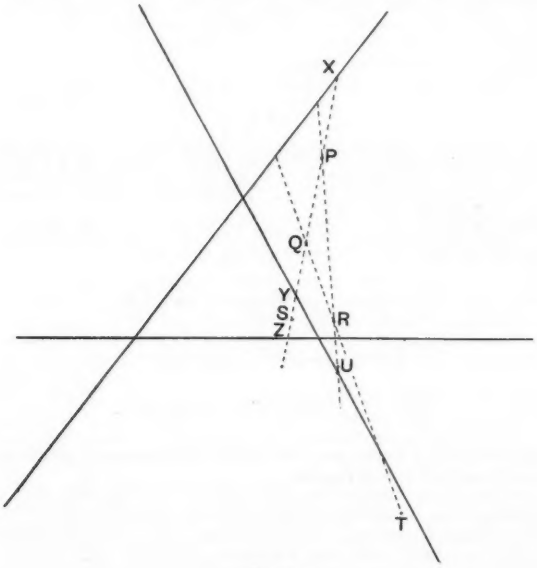

FIG. 2.

easily pricked out with the dividers, without drawing any straight lines, thus: to get S place the straight edge in alignment with PQ to cut the fixed lines in X, Y, Z. Transfer the distance PX, in its proper sense, along QY, so that X is transferred to Q: then if P' is the new position of P, $P'Y=PX+QY$. Transfer this distance, reversed, to Z along ZS: then $PX+QY+SZ=0$. With a little practice this can be done quickly, although the beginner will be liable to mistakes with signs. A true curve of the third degree will thus reveal itself, with the fixed straight lines for its asymptotes.

With beginners it would be well not to start with so general a case. All knowledge should proceed from the particular to the general; and if any attempt is ultimately to be made to classify cubic curves, it will be the easier when the special cases have come first.

I should take the fixed straight lines as forming an equilateral triangle; and I should assume that the three medians are axes of rectangular symmetry. This assumption is redundant: for our experimental elementary geometry will have shown that if a figure has two axes of rectangular symmetry not at right angles it must have a third: that these three meet in a point, and that they divide the four right angles there into six equal parts.

With these premises, *one* point will be sufficient to determine the curve. For a first case (Fig. 3) I should take this point, P, as lying without the triangle and within an exterior angle of the triangle. Three other points Q, R, S can then be determined as the symmetric correspondents of P with regard to the three medians. Place the parallels in alignment with BC; slide up till in

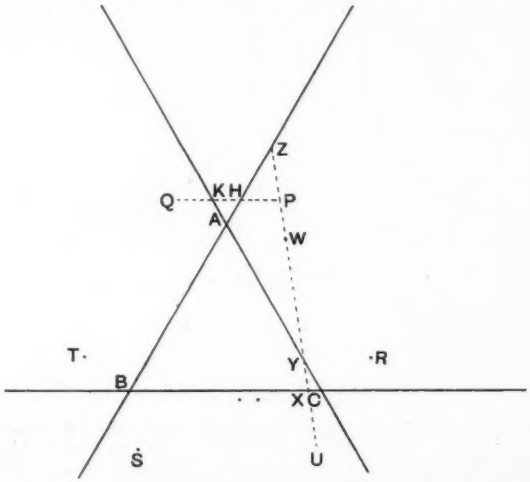

FIG. 3.

alignment with P. Prick off PH to AB: transfer to KQ: thus Q is obtained; and so R and S from the other two axes of symmetry. We can now find the symmetric correspondents, T and U, of R with regard to the medians through A and C (and in passing note that they are symmetric correspondents of Q and S). We thus have six points on the curve. Now join P to a non-correspondent point, U say, by a transversal cutting the sides of the triangle in X, Y, Z, and with the straight edge and dividers find W such that $ZP+XU+YW=0$. W determines five other points, just as P did; and as we add more points found, more will appear, and the curve thus revealed will be of such a form as Fig. 4.

A good deal of discussion can be promoted when so much of the curve is drawn. It can be seen that any line in general cuts the curve in three places, and there are reasons for concluding that no points in the curve can lie within the triangle. The continuity of the curve will also give opportunity for the development of geometrical ideas. In a preliminary discussion of any

kind of continuous curve which cuts a line in three places, P, Q, R, it will have occurred to any one that to pass continuously from P to R, through Q, one must pass through at least one point of contrary flexure—one must change from the inside to the outside edge. Now, in the figure, to go along the curve between P and Q is a plainly indicated path : to proceed from Q to R along the curve one must go to infinity in AC from C, back from infinity towards A from CA produced, round the whole portion of the curve lying within the angle C to infinity along CB produced, back from infinity towards C from BC produced, and so to R. As there are evidently no points of contrary flexure at a finite distance, if there are any they are at infinity ; and we have only visited *two* points at infinity : so we are to conceive of these two points at infinity as being points of contrary flexure; and if we continue

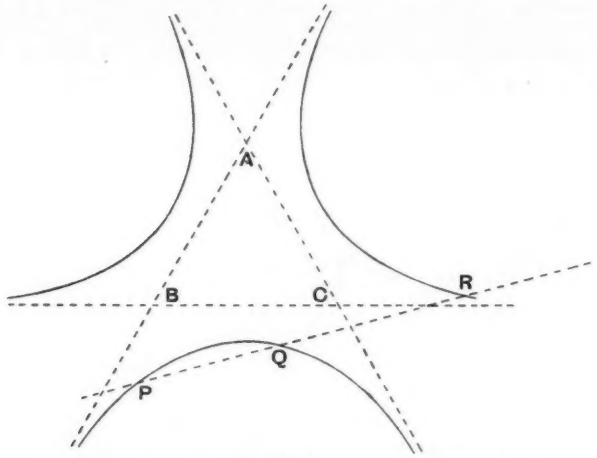

FIG. 4.

our path on from R till P is again reached, we visit a third point of contrary flexure, at infinity. So that three points of contrary flexure of this curve lie at infinity : and note that the curve returns from infinity *on the same side* of an asymptote as that on which it disappeared to infinity.

Such a curve might be called an escript equilateral cubic hyperbola.

Its equation, referred to oblique coordinate axes, might then be discussed. The axes being two of the asymptotes, and the sides of the equilateral triangle of unit length, it can be seen (Fig. 5) to be

$$xy(x+y-1)=k^3.$$

We can see that $KP=x$, $HP=y$, $LP=x+y-1$. Of these, as regards sign,

in space	1,	x is	+ ;	y,	+ ;	$x+y-1$,	− ;
„	2,	„	+ ;	„	+ ;	„	+ ;
„	3,	„	− ;	„	+ ;	„	+ ;
„	4,	„	− ;	„	+ ;	„	− ;
„	5,	„	− ;	„	− ;	„	− ;
„	6,	„	+ ;	„	− ;	„	− ;
„	7,	„	+ ;	„	− ;	„	+ .

Hence, since for the initial position, P, $xy(x+y-1)$ is positive, we see that k must be a positive quantity, and the curve lies entirely in the spaces 2, 4, and 6.

And when we put $x=0$, or $y=0$, or $x+y-1=0$, the resulting cubic for y (or for x) has three roots infinite: so that the three points at infinity along each asymptote are accounted for as being truly points of contrary flexure.

Finally, by choosing our initial point, P, nearer and nearer to a vertex of the triangle, we see how the curves obtained are associated with each other, and how ultimately they become merged into the sides of the triangle.

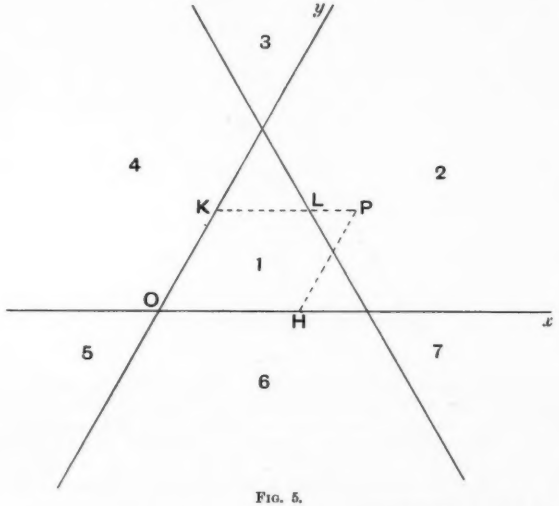

FIG. 5.

We can next essay to set out a curve in this way, giving the initial point within the triangle, or in compartment 3; probably the curiosity of the draughtsman—by now healthily aroused—will lead him to prefer the former. Having chosen a point within the triangle, and pricked in its symmetric correspondents, and theirs, he will find he has, as before, six points in the curve equidistant from the centroid (if the triangle is equilateral). The join of two non-correspondents being made, it will be found that the third point in this transversal will appear in space 3, 5, or 7; and the curve, as it develops, will appear to consist of three branches, each somewhat resembling half of a conic hyperbola, with a three-vertexed oval lying completely within the triangle. This will be the first instance of a curve with distinct branches not continuous. The three branches in spaces 3, 5, 7 may be regarded as one continuous curve passing three times across the line at infinity, having the sides of the triangle as asymptotes, and three points of contrary flexure thereon: and the oval is a separate curve. The whole might be called an inscript cubic hyperbola.

It will be noticed that if the initial point be taken near the boundary of the triangle, the oval and the hyperbolic branches tend to become merged into the sides of the triangle.

Again, if the initial point be taken nearer and nearer to the centroid, the

oval shrinks up indefinitely, the six points ultimately coinciding at the centroid. The oval has become a conjugate point; and it will not be difficult to assimilate the geometrical concept that every line through a conjugate point of a curve is a tangent there. The drawing of the inscript cubic hyperbola for this case will be very simple. For if P (Fig. 6) is a required point of the curve on a transversal XYZ through G, since there are two points of the

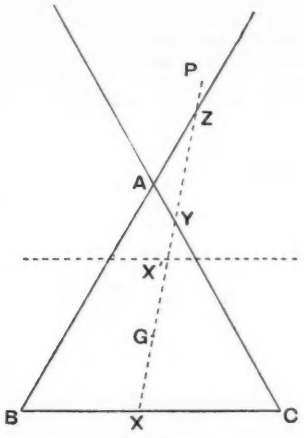

FIG. 6.

curve coincident at G, $GX+GY+PZ=0$. Draw, then, a parallel to BC, bisecting AG. Let XYZ cut this parallel at X': then, since $GX+GY=X'Y$, make $ZP=X'Y$. In particular there is a vertex, V, in GA produced, so that $AV=\frac{1}{2}GA$. The equation of the inscript equilateral cubic hyperbola is $xy(x+y-1)=-k^3$, where k is positive. If G, *i.e.* $(\frac{1}{3}, \frac{1}{3})$, is a point in the curve, $k=\frac{1}{3}$. And if k is greater than $\frac{1}{3}$, it can be seen that no portion of the curve lies within the triangle.

All these constructions can be carried out, step for step, if the sides of any isosceles or scalene triangle be used as asymptotes, there being a point of contrary flexure in each. The drawings will appear distorted, but the idea of oblique symmetry will aid the draughtsman to realise that the distortion is not anomalous, as he has used the medians as axes of oblique symmetry (without drawing them). The equation will be of the form $xy(lx+my-1)=k^3$. As an instance, an inscript isosceles cubic hyperbola might appear as (Fig. 7):

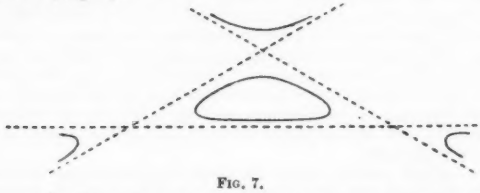

FIG. 7.

In this way the student will have practically drawn every kind of cubic which has three real asymptotes forming a triangle and three points of

contrary flexure at infinity—or, for brevity, we may say, three osculating asymptotes. In particular, the case in which the asymptotic triangle has shrunk up into a point will not have escaped his attention. He will now be able to draw in a speculative way other cubics having three real asymptotes. He may have realised that the existence of three axes of symmetry (oblique or otherwise) implies the presence of three points of contrary flexure at infinity.

He has developed a sense of symmetry which may have led him to discover that if any assemblage of points in a plane has two axes of oblique symmetry they must have a third, concurrent with the two others. A conic has an infinite number of axes of oblique symmetry.

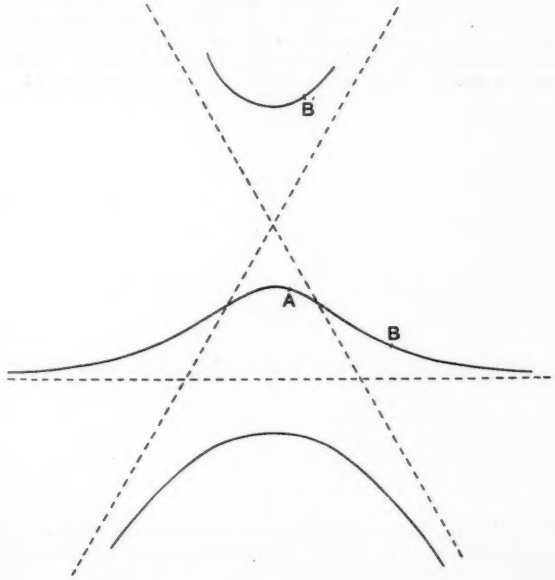

FIG. 8.

He will reduce therefore the number of axes of symmetry to one median of the asymptotic triangle, and may realise that the side which it bisects will have closer contact with the curve than the other two, and that there is a point of contrary flexure on the curve at infinity along this asymptote; *i.e.* that the curve has one osculating and two non-osculating asymptotes. This will lead him to expect the curve to disappear to infinity and reappear from the opposite direction *on the same side* of this asymptote. The other two asymptotes have the same kind of contact with the curve as for the conic hyperbola: so that the curve may be presumed to disappear to infinity along either of the non-osculating asymptotes and reappear on the *opposite* side of it. Naturally also, as every straight line cutting the curve in two real places must cut it again in a third, the cubic will cut each of these asymptotes at a finite distance.

The asymptotes being given, the number of points that may be arbitrarily chosen to determine the curve will now be enlarged from one to two. The consideration of the equation

$$xy(lx+my-1)+D(lx+my)+K=0$$

may serve to show this. The axes are ordinary asymptotes, the line $lx+my=1$ the osculating asymptote, and D and K arbitrary constants.

Experimental work will now show the various forms that these curves can assume. Incidentally it will be found that the method of construction used in the case of the three osculating asymptotes may break down. If for instance the two arbitrarily chosen points be taken (Fig. 8) as A and B, all the points obtained will lie on an independent branch of the curve touching the osculating asymptotes; and the other portion of the curve cannot thus be made to appear. But if A and B' be the chosen points, both parts of the curve can be drawn, and enough points set out to grasp its nature and discuss its properties qualitatively,* noting its three vertices. Or the curve may assume a form in which it has but one real vertex, as (Fig. 9):

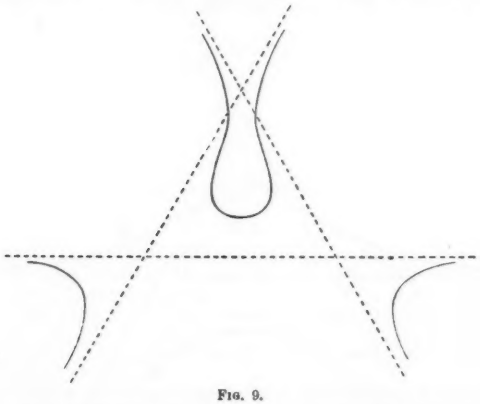

Fig. 9.

It will not be long before the draughtsman realises the existence of such a thing as a double point; which may be looked upon as the coincidence of two vertices, and lies upon the axis of symmetry. It will be easy to grasp the geometrical idea that every line through a double point cuts the curve in two coincident places there (and the two tangents there cut it in three). He will therefore look upon the arbitrary choice of a double point (on the axis of symmetry) as sufficient to determine the whole curve. The drawing will be then as much easier as was that of the inscript cubic hyperbola with the centroid as a conjugate point. For if D (Fig. 10) be a double point and any transversal be drawn through D cutting the asymptotes in X, Y, Z, the point P where it cuts the curve again will be such that $PX+DY+DZ=0$, since two points where it cuts the curve are coincident at D. And in particular the point where the curve cuts the axis of symmetry, which may be called the vertex, V, of the curve, is found by drawing DA to cut BC in X' and taking V so that $VX'=2AD$.

The shape of the curve will be found to depend on the position of D in the axis of symmetry. If D is within the triangle and less than a third of the

* The independent portion of the curve may appear as an oval within the triangle.

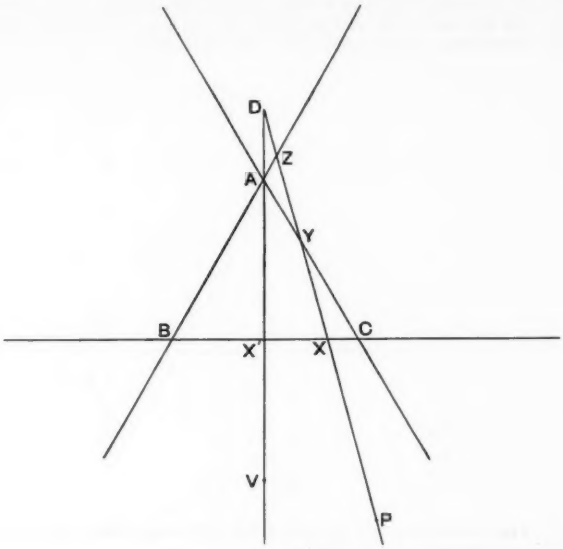

FIG. 10.

way along the median from the vertex A, there will be a loop with a definite area, as (Fig. 11):

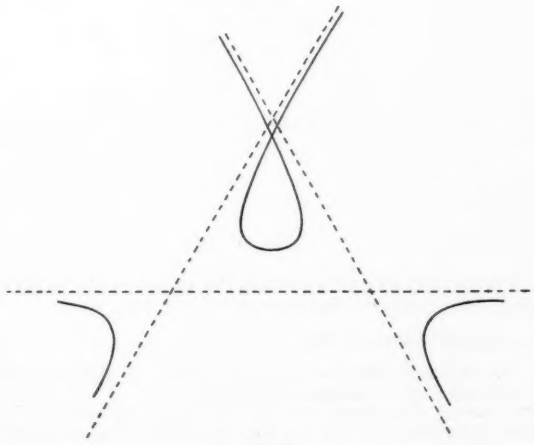

FIG. 11.

If D is taken further from the vertex, the loop diminishes; until, when D is a third of the way along the median (Fig. 12), the loop has shrunk to a point and we have a cusp—three vertices coinciding there.

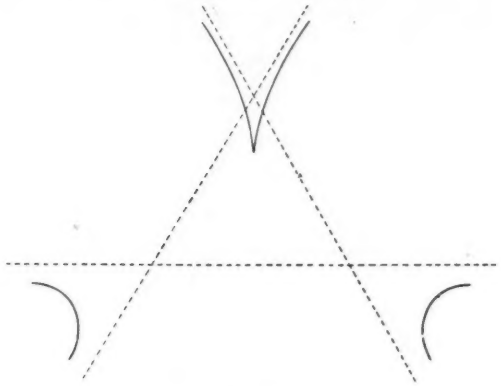

FIG. 12.

Further on, D becomes a conjugate point, still comprising two vertices, the third being nearer to A (Fig. 13):

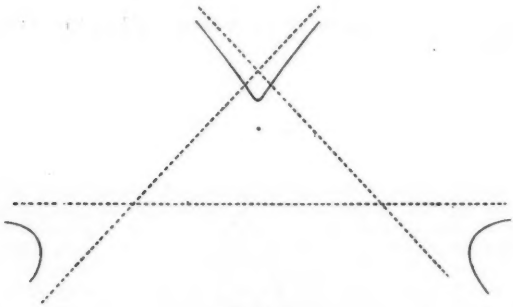

FIG. 13.

When D is half-way along the median, the vertex of the curve is at the intersection of the non-osculating asymptotes (Fig. 14).

Further still, the form is as Fig. 15.

When D is at the centroid we have the special inscript cubic hyperbola before mentioned; all the asymptotes osculate.

If D is between the centroid and the base of the asymptotic triangle, the other points of intersection with the non-osculating asymptotes appear below the base, as Fig. 16.

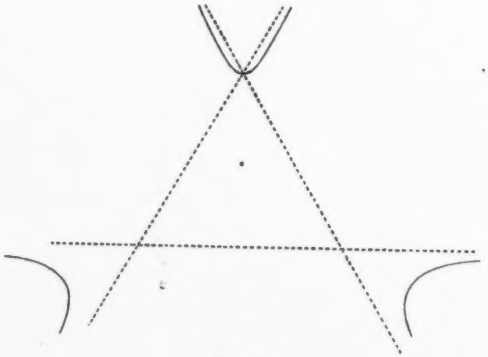

Fig. 14.

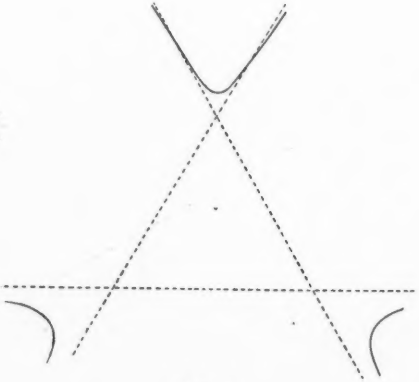

Fig. 15.

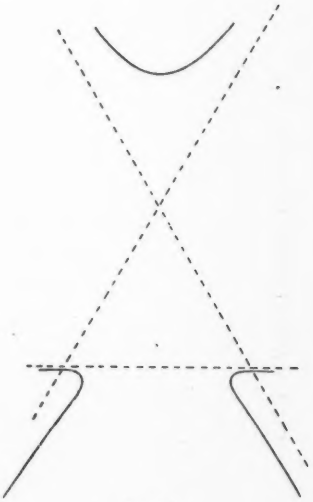

Fig. 16.

When D is at the foot of the median, the cubic breaks up into an ordinary conic hyperbola with a tangent thereto. Finally, if D be taken in the median produced, we again have an actual double point there, as Fig. 17 or Fig. 18, on page 287.

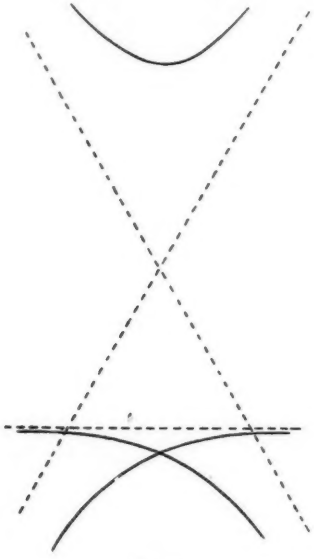

FIG. 17.

I have now sketched as briefly as I can the way in which I think third-degree curves might be approached at an earlier stage of a young geometer's education than is usual. The whole subject covers a very wide field, but perhaps this might do for a beginning, confined as it is to the drawing of cubics with three real asymptotes, all given, and only some cases of these. Whatever misgivings one may have as to the possibility of acquiring marks by this study, the time spent on it would be happily and engrossingly passed. It develops many good mental and practical qualities. Careful draughtsmanship, which, so far as I know, receives recognition as a mathematical asset only in the London B.Sc. Pass Examination; clear and logical ideas upon the infinite in planar space, which must react in the larger study of space of three dimensions and in the ultimate philosophy of geometry; the habit of speculating, of making records, of classifying for one's self, are all encouraged. And all rests upon one theorem, easily proved, inferred at the outset of Newton's "Enumeratio," which may be thus stated:

"*Co-asymptotic cubics are co-diametrical.*"

R. W. K. EDWARDS.

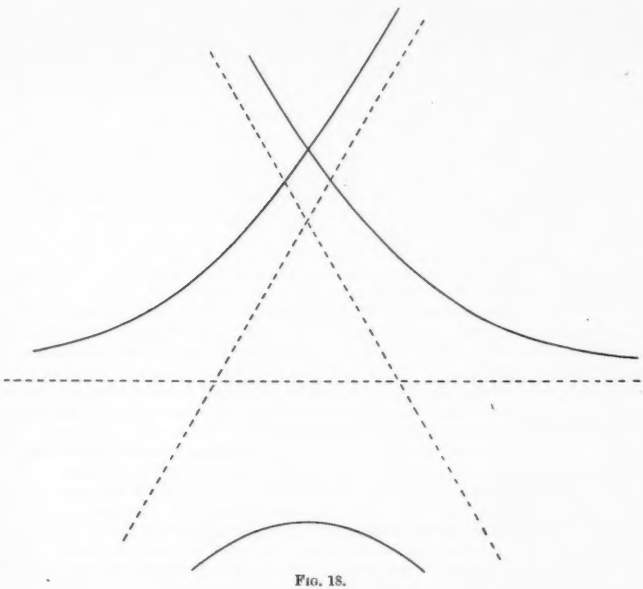

FIG. 18.

ON PRACTICAL MATHEMATICS IN SCHOOLS.

WHEN, with the other members, I received a request from the Secretary, it seemed only fair to him that I should express my willingness to do whatever might be in my power towards the provision of a subject for discussion; and therefore, with an earnest wish that he would if possible ignore my suggestions, I proposed the rather well worn subject of this paper. To my dismay he did not ignore the suggestions, and in consequence you will have to listen to the trite observations of one who, sufficiently remote from great centres of education, and in great measure precluded from intimate contact with modern methods, is nevertheless sufficiently interested in them to form theories, or rather opinions, of his own. Those to whom the element of novelty may be absent in the points that I proceed to consider will, I trust, extend a charitable reception on the great moral principle of the Western States that the pianist who does his best is not to be shot.

In the ordinary course of lecturing to students who come stamped with the approval of some of the entrance examination Boards, and their name is legion, I find that, especially when they have had a more or less practical course, very great difficulties arise on account of their weak grasp of mathematical reasoning.

The consideration of the various elements that constitute "practical mathematics" that were thus pressed upon my attention has led me to certain conclusions which will be fairly clear if I proceed to take the headings of the short syllabus in your hands.

(i) *Beginners—Simple Tests and Measurements.*—As far as I know some of the simplest measurements practised by beginners are made on squared paper; and are concerned with rectangular and other rectilinear figures, circles, irregular areas. Some time seems to be wasted over this kind of work at an early age; and I feel some doubt as to its real value until the pupil ceases to be a mere beginner. The measuring of a rectangle, integral sides, by counting squares, is of course an obvious and fundamental experiment; but if the sides are fractional the method may be misleading, and is quite valueless unless the student understands the reason why the product rule applies in all cases; if he does understand this rule one experimental illustration, not "proof" exactly, is as good as a hundred.

As to circles, it must be remembered that we have to give an "average" value to incomplete squares, but that this "average" may depart in certain cases materially from one half: as in that of a rectangle, for instance, of $4{\cdot}87 \times 3{\cdot}78$ inches. This idea of an average is not quite so simple as it seems; and although when a pupil knows a little algebra I should make no objection to this circular measurement, yet for beginners I think the game is hardly worth the candle.

What is true of circles applies equally to other curved figures, and to spend time in a mere enumeration which can only give a result incapable of explanation is not to teach mathematics. Of course, if some independent verification can be used, the circumstances are entirely changed.

Measurement by weighing is probably mere waste of time. It is a good enough practical method for readily and quickly finding complicated areas, but it teaches nothing, and moreover assumes that the material used is of uniform thickness. It is scarcely mathematics at all.

But there are certain measurements, practical measurements, which drive home and enforce principles: these are, I think, of the highest value. For example, to make a right angle by folding a straight-edged sheet of paper affords an illuminating illustration of what a right angle is; the verification of a set square by reversal on a straight edge is another rigorous and logical test; the bisecting of angles by mere folding, and the verification in the same way of the symmetry of a circle, together with its subdivision, are all, I think, simple, instructive and logical exercises.

Again, the method called "dissection," which, of course, does not necessitate actual cutting up, a process of which the utility depends on the quality and stage of the pupil, may be illustrated in a most interesting way by means of the actually dissected figures; one of the peculiarities of this method being that the actual and accurate process of dissection if often much briefer than the formal statements to which it can, of course, be reduced.

(ii) *Relations between Geometry and Algebra.*—These are all-important, but it has always seemed to me that before any of the formulae by which it is the fashion now to "prove" Euclid II. can be used advantageously, strict and full proofs, for rational quantities, of the formulae for the areas of a rectangle and square should be given. An extension to the solid is easy, and these ideas being mastered, an important and intimate relation between geometry and algebra is established. From this relationship a vast number of useful and practical applications can be made and tested by actual construction. For while I think there is very little good gained by setting a child to measure quantities which once obtained have no relation to anything else, yet an algebraic result (*e.g.* length of shortest chord through a given point within a circle) when verified by actual and fairly careful construction has a real value due to the association of the methods.

(iii) *Algebraic Verifications.*—Far too little is made of this; and there is no doubt, I think, that the general ignorance as to the kind of values assumed by algebraic expressions is extraordinary. It is of the utmost importance that pupils should be able to evaluate algebraic expressions readily, and should be able to verify such formulae as

$$x^4+4a^4=(x^2-2ax+2a^2)(x^2+2ax+2a^2)$$

for small values of x and a with rapidity and accuracy. This kind of work brings a general knowledge of the powers of the smaller integers, and the rapidity with which an exponential expression increases.

So far for the few comments I have to make on the beginners. With the older pupils I am on more familiar ground; and with them I shall occupy your attention for the rest of my time.

(iv) *The Plotting of Curves by Points.*—I do not think sufficient care is taken, as a rule, to point out that no mere calculation and marking of points can by itself give any exact knowledge as to the shape of a curve. I venture to think that in mathematics pupils should not only be told this, but should also be told that the simple kind of curves that they are likely to meet with will be smooth and without sharp peaks of any kind. It is surely better that pupils should provisionally accept this statement than that they should, from the various well-known causes that affect them, produce the monstrosities that often appear. A very good kind of curve to give is

$$y=x+\frac{x(x-1)(x-2)(x-3)(x-4)}{24}\equiv\frac{x^2(5-x)(x^2-5x+10)}{24};$$

it is easily traced, and offers an opportunity for explanation of a valuable kind.

When pupils have traced a few curves "by points," they may with great advantage and without much difficulty be shown how they can often make certain deductions from the equation itself, and thus save much needless calculation. A pupil naturally gives x a series of values in arithmetical progression, and equally naturally some element in the curve is thereby made obscure. A little study of the equation will often show how to choose a more convenient system of x values. It is perhaps unnecessary to illustrate this, but one example may be given: consider the curve $x^2+xy+y^2=1$; here its symmetry can be seen from the interchangeability of x and y, and change of sign of both. This shows that only one properly chosen quarter need be constructed. Again, $3y^2 \not> 4$, $3x^2 \not> 4$, which limits the curve, and the rest is fairly simple.

In many cases I see that students are allowed to take two different scales for x and y; this certainly does not give the curve sought: and, I think, that when such a permission is given, it should be explained that what we get is a projection of the actual curve. Also, pupils very properly might be taught that when for small values (or large) of x and y there is some doubt of the form, a new drawing of that portion of the curve on a larger (or smaller) scale should be made. I have actually found this done with much intelligence in several cases.

To the mere marking in of points calculated without some consideration of interval and so on, I attach little value beyond what naturally appertains to any careful and accurate numerical work; to the systematic selection of points, and to the use of suitable methods of computation, I attach a great deal.

Of recent years a good deal of importance seems to have been assigned to

(v) *Graphic Solution of Problems.*—The kind of problem perhaps most suited to this treatment is that which deals with travellers or

trains overtaking or passing under various conditions. To those who really understand what they are doing the graphic solution is valuable and interesting—not so much for the result which can almost always be obtained more easily by analysis, but for the continuous picture the method affords of the manner in which the intervening stages to the final result are passed. It does not seem that much attention is given to this side of the mattter; and I should be rather surprised to learn that even the fact that this method affords a continuous representation of something real is usually grasped by the student.

And, as I am referring here to realities, I may perhaps be allowed to digress for one moment on a very general point: *analysis*, by which broadly I mean algebraic manipulation, has been a good deal discouraged of late years, with results that have proved disastrous in the early stages of the differential calculus. In order that the ideas of the calculus, new and difficult to the beginner, may be taught properly, it seems to me that there should be not only plenty of functional material, so to speak, from which to rear the new structure, but also facility in dealing with it. The extraordinary ineptitude displayed by the junior honours student in working out fundamental derivatives seems to be due to his very imperfect acquaintance with the simplest transformations of algebra and trigonometry; and this in turn seems to be due to his habit of according a half-hearted intellectual assent which has not been compelled by abundance of illuminating practice of simple problems in analysis. To expect a useful and active comprehension of algebraic principles, however clearly each in a dull progression of logical conviction may be accepted, is as futile as to expect a violinist to play effectively because he knows exactly how to produce, given sufficient time, any particular tone that may be required; or to hope for a graceful Tango from the person who knows each movement by heart, but has never performed one of them. For these and similar reasons I would urge a return in some measure to the old exercises in transformation and analysis, with one important extension—that as far as possible students should try to interpret their formulae.

This is particularly the case in coordinate geometry, which, a simple and beautiful combination of number and form, is too often a dreary wilderness to those who have no clue to its system. May I illustrate what I mean by an example? Instead of the beginner being only talked to about equations, etc., let him be asked to mark on a squared sheet the values of $x+3y+1$ at many points, and let him then notice (1) where the values are constant, (2) on what their variation depends. He will easily realise the geometrical value of the expression $x+3y+1$, and will thence deduce the meaning of the conditional equation:

$$x+3y+1=0.$$

The same valuable exercise may be set with regard to such expressions as $xy-1$, x^2+y^2, and so forth; and we have an indefinite field for practical exercises of a most illuminating kind.

The next point on which there seems occasion for a few remarks is

(vi) *Computation.*—A good deal of interest attaches to correct numerical work in connection with the binomial theorem, the verification of such identities as those obtained for the expansions of e^n—*e.g.* $e^2 \times e^{-2}=1$, and of various logarithmic identities. Now, apart from logarithmic work, of which I shall speak immediately, the art of direct computation is by no means despicable, and I only wish that my friend Mr. Langley's little book on this subject were more widely known. In no single matter are schoolboys worse informed than in the methods of computation. How will they usually work out a series either binomial

or exponential? Instead of calculating each term from the previous term with possibly some modification, they will reduce each term to a vulgar fraction, probably making mistakes on the way, and will then reduce these fractions to decimals, in which process they will certainly make more mistakes; they will be asked for a result to four decimal places, and having no idea where to stop, will produce one correct to two or to six. When they come to the square-root process, this should never be taught without at least the statement which I myself think should be amplified into the extremely simple proof, that at every stage the square of the number obtained has been taken from the number given, while the remainder is shown in the work. Probably most people will agree, but I may assure those present that not one of the students of a certain university class who had studied "square root" at school had ever been told this elementary fact.

No one seems to grudge the time spent over more or less feeble "practical questions" about carpets, walls, flowing water, etc., but could not a little time be given with great advantage when "square root" is being studied to the actual verification of the principle quoted?

As to such verifications as

$$\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5.5^5}+\ldots+\frac{1}{7}+\frac{1}{3.7^3}+\frac{1}{5.7^5}+\ldots=\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\ldots,$$

are not these sufficiently interesting and practical to demand a share of the school time?

And now I approach a more seriously disputed point.

(vii) *The Study and Use of Logarithms.*—I shall begin with three facts of recent experience. The first is, that the question being asked "how will you most quickly evaluate $\frac{1}{1.16}$ to six decimal places?" in connection with a binomial problem, the answer expected was "divide ·1111... by 2 and by 7," or "divide ·055... by 7." The answer given was "by logarithms." The second is, that in a public examination where logarithms were allowed, 12×13 occurring in a question was found frequently "by logarithms" and $6\times(1{\cdot}05)^2$ was obtained by finding $(1{\cdot}05)^2$ by logarithms and then multiplying by 6; the third case was of an advanced student in another public examination, where 4-place tables were provided, complaining that he was not used to tables that had no antilogarithms.

Now, although much may be said as to the culture involved in a knowledge of the theory, very little can be said in favour of teaching the mere practice of an isolated method to those who will never use it. How many persons in an average school even in its higher classes will ever have occasion to make real use of logarithmic tables? Only those who are to be engineers or actuaries; and yet, for the sake of these, the modern outcry for practical methods has led to the inclusion of silly and pointless sums to be done by unintelligent and blindfolded application of certain mnemonic rules accompanied, if success is desired, by a nimble thaumaturgy exercised in rapid turning of pages and unearthing of entries.

And because some philosophers have said that unless you use a real table the exercise is useless, complete success in many examinations depends not on intelligence, not on ability, but on skill in using a kind of debased lexicon.

For the great majority of students the index theory of logarithms is the thing which matters, and any necessary practical illustrations can be best provided for either by giving the needed logarithms, or by giving the reprinted page, or portion, of the tables that may be required. This procedure, I am informed, has already been followed in certain

examinations, and I trust the example will be contagious; on the other hand, I trust those questions in which a nervous, shortsighted or awkward pupil, by reading a "natural" instead of a "logarithmic" tangent, or by dropping a single figure out of a dozen in his process of extraction may thereby destroy a long piece of otherwise excellent work, will not for long disfigure the practical examination paper.

On the other hand, by judicious study, the ordinary tables may be made both instructive and interesting if a little care is taken to use them in the right way.

For example, a most cursory glance at them shows that the differences change; to ask for a graph connecting the difference and the number and to discuss the law by study of the tables is surely an excellent exercise; the method applied to a sine table is very interesting in its result: the use of the tables in plotting such expressions as

$$2 \sin\theta \sin 3\theta \equiv \cos 2\theta - \cos 4\theta$$

throws considerable light on the turning points of this function and so forth.

These are illustrations of an educative use of tables; and in this I have a belief as strong as my disbelief in the utility of making the average pupil practise the actual page turning of some bulky logarithmic table, an act which he will in nineteen cases out of twenty never require in his future life, and which in the twentieth case can be acquired more readily in a week of actual experience in an office with the particular kind of tables used than with six months of assiduous practice at school on the general lines that must necessarily be there followed.

From this abstract subject I turn to the elegant and much cultivated art of

(viii) *Linkage Description.*—Most of the linkages in ordinary use are of extreme complexity when treated analytically, and therefore, except as objects of intelligent but unsatisfied curiosity, there is not much to be said about them. To me the very beautiful elaborations based upon Kempe's studies and the investigations of Hart, Cayley and Peaucellier are intensely irritating, because, at least I find it so, one cannot in a moment see the cogency of the result.

On the other hand, the pantagraph and the Peaucellier cell and an angle trisector should be in every higher grade school, with possibly the addition of the simple three-bar appliance, Hart's inverting linkage, and a model of Cayley's linkage on account of its unexpected "porism," if I may use the phrase.*

Mr. Langley knows more about this subject on its pedagogic side certainly than most of us, and he has shown me admirable work done by the boys at Bedford under his instruction. It is so extremely easy to get fairly satisfactory results from home-made appliances that it seems quite worth while to encourage every boy in a higher science class to make one or two of these for himself.

I come next to a relatively small division, namely,

(ix) *Models of Surfaces and Curves.*—The number of these suitable for school purposes is not great; nevertheless, I think that such models, if explained in general terms, are of sufficient interest to encourage further study. Many of them are probably best obtained from the manufacturers, but others can be quite well made at home or at school. The figures of the second degree as built up by circular sections are best

*From a model carefully planned by me and still in my possession, Mr. Hilger has made excellent copies, which he has placed on the market.

bought; the string models of the cone, the paraboloid and the hyperboloid of revolution can be easily made; while the ruled models made of wire are best bought. All three classes are intensely interesting, and if to them Crum Brown's cubic curve is added, a very stimulating source of curiosity will be found.

The regular solids, the star-shaped regular solids, the tetrakaidekahedron of Lord Kelvin are more obvious and simple examples.

So far I have given little more than a catalogue of subjects which require practical treatment—now for the conclusion, which will be extremely short.

I have advocated concrete and practical treatment as companion to theory for more years than I care to name; and I remember more than 20 years ago receiving severe criticism from a schoolmaster because, in examining a school, I tried to impress upon the pupils that a pint was really so many cubic inches, and that liquids could be, and in France were, sold by cubic measure! and by another schoolmaster because I suggested that to ask small children to multiply 9532786421927 by 3257 was neither so useful nor so interesting as to invite them to find the cost of 147 pairs of boots at 11s. 6d. I remember, too, that an important examining board looked with extreme disfavour on this deduction: "A point O is 13 inches from the centre of a circle, radius 5 inches, OPQ cuts the circle in P, Q, so that OP is 9 inches; find the length of PQ." This was far too numerical.

These I give to show that I am by no means opposed to "practical" questions.

But when the practical problem which should elucidate the intellectual theorem in which the true culture of mathematical study lies (for mathematical study from arithmetic to complex theory, from Euclid, dear old name, to polytopic geometry, from the numeration table to ideal primes is a great culture), when I say this "practical" work is taken as the only object and end of all mathematical study, I feel that it is time that somebody, or better still Some Body, should cry halt and consider whether we are not in danger of purchasing our practical mathematics at too great a price. Already the agitation, attacked so well in the early days of its recent reincarnation by Lewis Carroll in his *Euclid and his Modern Rivals*, against Euclid is showing signs of disintegration, and already the intuition of "direction," as explained by sliding a set square along a straight edge, is being discredited; for no longer does everyone admit that the statement of Euclid and Playfair that "through a point outside a line only one line parallel to it can be drawn," is more dogmatic or less lucid than the following modern statement: "Lines which intersect have different direction; lines which have the same direction do not meet, and are parallel"!

Already more than one earnest and successful teacher of boys is looking back with uncertainty—changing to disappointment—as to the results of the modern theory of school mathematics.

Mr. Strong, Rector of the Montrose Academy, is well known as an experienced and successful teacher and administrator; he has always taken a special interest in practical mathematics, and he very kindly wrote me a valuable letter upon his experience in this branch of study. His opportunities for forming a just estimate are exceptionally favourable, and he states the case with a clearness and cogency on which I could not hope to improve; these are his exact words:

"I think that perhaps you would be interested in one or two conclusions we have been forced to form over our experience here. There is no question but that the pupils are interested in the practical part of

their work in Mathematics, but unless the thing is watched carefully there is the danger of a great waste of time. In any case, so far as the Mathematics hour is devoted to practical work, it is not all gain. Such gain as there is, is largely to the interests of the middle and tail end of the class. Those of our men who have worked the old and new methods, and are therefore in a position to compare, think that the best pupils at the age when they leave school do not reach the standard which the best ones reached under the old system. On the other hand, the new system has made the doubtful pupils, *i.e.* pupils on the border line of a pass, more sure of reaching a pass standard. There has been a levelling process at both ends. . . .

"One other word, after the anarchy following the deposition of Euclid, there seems to be a tendency among teachers just now to hark back again to *authority*. Our men, for instance, feel that there would be a distinct gain to Geometry teaching in the general adoption of some definite system, a standard course, so to speak. As things are, there are too many opportunities for loose thinking."

These experiences and opinions are entirely borne out by observation on my own students, who, coming almost always from Scottish High Schools, Grammar Schools, and Academies, enter the University and continue their mathematical course there.

In conclusion, let me express a hope that the authorities of the Mathematical Association will keep in the future, as they have done in the past, a firm faith that the primary value of mathematical study is not to be found in such results as skill with the penny ruler, the scissors and the scales, but in the culture that all genuine mathematical study has been held to give from the days of Plato to those of Russell and Whitehead.

J. E. A. STEGGALL.

MATHEMATICAL ASSOCIATION.

GEOGRAPHICAL DISTRIBUTION OF MEMBERS.

A LIST of members of the Association on January 1st, 1914, classified according to locality, shows a total of 747, of whom England contains 582, Wales 28, Scotland 33, Ireland 13, Isle of Man 1, Channel Islands 2, South Africa 25, West Africa 2, Canada 7, Australia 12, New Zealand 5, West Indies 1, Malta 1, India 15, Straits Settlements 1, Brazil 1, China 1, Egypt 6, Germany 2, Hungary 1, Italy 1, Japan 3, Roumania 1, Russia 2, Syria 1.

The 582 members in England were distributed thus: London 148, Beds 10, Berks 12, Bucks 12, Cambs 28, Cheshire 12, Cornwall 1, Cumberland 5, Derbyshire 5, Devonshire 12, Dorset 1, Durham 9, Essex 13, Glos 33, Hants 13, Herts 14, Hereford 2, Isle of Wight 3, Kent 27, Lancs 33, Leicester 3, Lincs 2, Middlesex 7, Monmouth 1, Norfolk 7, Northants 3, Northumberland 9, Notts 5, Oxfordshire 14, Rutland 4, Shropshire 5, Somerset 9, Staffs 7, Suffolk 3, Surrey 26, Sussex 14, Warwickshire 15, Westmoreland 1, Wilts 4, Worcs 8, Yorkshire 42.

H. D. ELLIS.

LONDON BRANCH.

THE annual meeting of the branch was held at the Regent Street Polytechnic on Saturday, February 7th.

The annual report of the committee and the financial statement of the treasurer were presented and approved.

Dr. F. W. Dyson, F.R.S., Astronomer-Royal, was elected president for the year 1914.

Mr. Whipple was elected Chairman; Professor Nunn and the retiring Chairman, Mr. Abbott, were elected Vice-Chairmen. The Treasurer and Secretaries were re-elected.

Dr. Dyson delivered an address dealing with the uses that might be made of astronomical facts in teaching elementary mathematics and giving some account of recent astronomical research.

Mr. Abbott read a paper on "Practical Work in Elementary Mathematics." This was illustrated by models actually made by boys in the Polytechnic School. There was also an exhibit of models and apparatus in the Mathematical Laboratory. A discussion followed, in which Mr. Daniell, Professor Bickerton, Mr. Boon, and Mr. Ellis took part. Mr. R. F. Davis distributed some graphed copies of useful geometrical illustrations of a formula in differential calculus. It is hoped that other members will follow this excellent example.

CORRESPONDENCE.

THE EDITOR OF THE *Mathematical Gazette*.

DEAR SIR,—Is it not time that public opinion should refuse any longer to permit the time-honoured laxity in the use of the words "because" and "therefore" in mathematical text-books?

Thus, in books on elementary geometry, the proof that the opposite angles of a parallelogram $ABCD$ are equal is usually given somewhat as follows. Join BD. Then *because* BD meets the parallels AB and DC, the angles ABD and CDB are equal. Again, *because* BD meets the parallels AD and BC, the angles CBD and ADB are equal. *Therefore* the whole angle at B equals the whole angle at D.

This definitely asserts that the angles B and D are equal because BD is drawn, while presumably we merely wish to say that we are able to detect the equality of the angles because we have drawn BD.

To take another illustration, here is a passage from a book on algebra (the best with which I am acquainted): "By the square-root process we can show that $\sqrt{4{\cdot}053} = 2{\cdot}0132\ldots$;

$$\therefore\ 2{\cdot}0132 < \sqrt{\sqrt{5} + \sqrt[3]{6}}."$$

It would surely be better to avoid the statement that one number is smaller than another because we can use the square-root process.

While instances as marked as these are to be found everywhere, still more numerous are statements which, though less strikingly indiscreet, are yet questionable.

Thus, in the course of a proof we are hardly entitled to say that two triangles are congruent *because* we find certain parts in one equal to certain parts in the other, and would do better to content ourselves with saying that we are aware of their congruence on that account; and yet it is the first assertion that is almost invariably found even when an alternative proof on the opposite page asserts equally firmly that something else *causes* the triangles to be equal.

Another tolerated statement which is equally unhappy is that one proposition or one property of a figure *depends* upon another, when it is merely meant that *one proof* of the one depends on the other, since the two propositions if true at all are both true simultaneously, and are co-equal, neither coming before or after the other.

One possibility would be to state clearly that in mathematics "because" and "therefore" are to be unemphatic, and to refer to our processes of perception, not to the facts perceived, but this seems undesirable. If, for

instance, we have to prove something about a square $ABCD$, we may (I suppose) assert with any emphasis we like that *because* $ABCD$ is a square, *therefore* $AB=BC$, or if x is defined as $=2$, we may say $\therefore\ x^2=4$ without restricting $\therefore$ to mean "therefore we perceive that."

The only other alternative would seem to be to deny ourselves the conventional laxity and to write "hence we see" instead of "therefore" wherever the latter word is too strong a conjunction.

Even in such an argument as

$$6x=30,\quad \therefore\ 3x=15,\quad \therefore\ x=5,$$

the $\therefore$'s appear rather to overdo it, for we might equally well have

$$6x=30,\quad \therefore\ x=5,\quad \therefore\ 3x=15,$$

and consequently the rule of self-denial suggested would be tiresome. Does not, however, the spirit of the time demand it?—Yours, etc.,

CHARLES HARDINGHAM.

7 Queen Anne Terrace, Cambridge,
February 2nd, 1914.

The Editor, *Mathematical Gazette.*

DEAR SIR,—In your December issue you published a notice of my book on "School Algebra." I should be very grateful if you would allow me the favour of a reply to the more important criticisms.

Your reviewer's main criticism is that my treatment of algebraic theory does not conform to the requirements of formal Algebra. I would submit that the methods of formal Algebra, in which, for example, the rules of signs are matters of definition, are not suitable for beginners, and that the problem which confronts the writer of an elementary text-book is to devise suitable explanations of algebraic processes based in the first place on arithmetical conceptions.

The rules of signs can be demonstrated without difficulty in the case of algebraical expressions in which all the letters represent positive quantities and all the signs have their strictly arithmetical meaning. Presumably this was how the rules were first derived. In any case I think this line of advance is the correct one for teaching purposes, and for this reason practically the whole of my theoretical work in Chapters I.–XIII. assumes that the letters represent positive quantities, and that the minus sign indicates arithmetical subtraction: §§ 19-22 are a digression, as I have indicated in § 21.

In Chapter XIV. I am not dealing with the negative numbers of formal Algebra, but with negative quantities such as occur in Mechanics, Trigonometry and Coordinate Geometry.

The fact that the rules of signs are matters of definition in pure Algebra implies that the negative numbers of Algebra may not be used to represent the negative quantities of other branches of mathematics until we have verified that these negative quantities will conform to the rules of signs.—I am, dear Sir, yours faithfully,

A. G. CRACKNELL.

I cannot agree with Mr. Cracknell that the methods of formal Algebra are not suitable for beginners: the method of presentation is, I believe, a question of taste simply, and was not to any great extent in my mind when reading the "School Algebra."

Mr. Cracknell professes to base his reasoning on arithmetical conceptions. So far, very good. But his arithmetical conceptions include many algebraical

ideas. Thus, on page 173 it is stated that "$(+3)+(-7)+(-6)+(+2)$ *corresponds* exactly to the arithmetical process

$$+3-7-6+2=+5-13=-8."$$

There is no arithmetical process here, for the first step $+3-7$ is impossible.

Similarly, on page 94, we have "$x^2=\frac{3}{4}\times\frac{3}{4}=\frac{9}{16}$" (presumably quoting an "Arithmetical Rule"): this is a very good "rule," but unsatisfactory as a foundation for the "Laws of Algebraical Fractions." For this rule cannot be proved or even "explained." What would be the result of sending a boy to a shop for 6 lbs. of sugar $1\frac{1}{2}$ times?

If, as Mr. Cracknell says, Chapter XIV. only refers to quantities such as occur in Mechanics, Trigonometry and Coordinate Geometry, the note at the foot of page 173, where it is stated that "the signs inside the brackets indicate positive and negative *numbers*," should be deleted as misleading. Besides, what is the use of this chapter, for no operations are performed by negative quantities? The sign of a "moment" does not materialise out of the sign of "the force" and "the distance," but is added afterwards by a totally distinct consideration, at any rate in Elementary work: nor does a conscientious teacher allow the usual definition, "moment = force × distance," to pass without an emphatic warning that only the *numbers* representing the measures are concerned. If so, we should have 3 oranges × 2 apples = 6 orange-apples. In Coordinate Geometry all the letters stand for numbers, or else x^4 has no meaning. All that is required in the way of interpretation is the meaning to be attached to the negative sign as a sign of *quality*: the whole of the calculation is concerned with abstract numbers alone.

Even the operation of counting *things* has no relation to the things counted. There are two mental processes going on simultaneously, one noting the thing and distinguishing it, the other counting: and these are independent.

Similarly, if we require the result of three twists, two right-handed and one left-handed, whose magnitudes are 20°, 30°, 60° respectively; we may *interpret* a right-handed twist as positive and the left-handed as negative, or *vice versa*. The mind makes a note of this and of the unit employed, but the calculation is with abstract numbers—either

$$(+20)+(+30)+(-60)=(-10) \quad \text{or} \quad (-20)+(-30)+(+60)=+10$$

—and the result in each case is interpreted as a *left-handed-twist in degrees* (supplied by the mental note), whose magnitude is *10* (supplied by the calculation). I think the main points of my criticism are founded on the wide divergence between the conceptions held by Mr. Cracknell and myself as to the boundary line between Arithmetic and Algebra: and I hold that if he assumes that a boy is "familiar with" such a result as $3-7+5=8-7$ in Arithmetic, he must *prove* that this is justifiable before he founds a theory of Algebra on it: similarly, for fractions it is necessary to *prove* that in $\frac{3}{4}$, as a multiplier, the 3 and the 4 are separable, and much more, before it can be used as a foundation for an Algebra that is to be convincing. My opinion is that Arithmetic is concerned only with positive integers, whether expressed numerically or literally, as far as theoretical processes are concerned.

X. Y. Z.

7 Queen Anne Terrace, Cambridge,
February 12*th*, 1914.

The Editor, *Mathematical Gazette*.

Dear Sir,—I thank you very much for your kindness in allowing me not only to reply to your reader's criticisms, but also to see his reply to mine.

With regard to the second paragraph in his reply, I would point out that there is a strictly arithmetical interpretation of the equation $+3-7=-4$,

viz. that to add 3 to any number (not less than 4) and then subtract 7 gives the same result as to subtract 4 from that number.

As for the statement $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$—if in dealing with beginners I am not allowed to assume it from arithmetic, nor to explain it, nor to prove it except by the rigorous treatment of the Theory of Number, I fear I must abandon the unfortunate equation in despair.

For the rest I should like to discuss briefly the application of + and − signs in connection with the moments of forces about points.

If two forces have moments (+3) and (−7), we apply the rule of signs for algebraical addition, and infer that their combined moment is (−4). To justify this it would not be sufficient to say that the + and − signs are convenient symbols for counter-clockwise and clockwise. It would be sufficient to show from dynamical considerations that the two moments we are calling (+1) and (−1) when combined give a resulting moment of zero: for then moment (+3) cancels with moment (−3) and leaves moment (−4). Extending this idea, I have shown in § 119 that where a negative unit can be defined as that which cancels a positive unit, the quantities concerned will obey the rules of signs for addition and subtraction.

Again, we may apply the rule of signs for multiplication in the formula:—

$$\text{moment} = \text{force} \times \text{arm}.$$

To justify this it is not sufficient to say that the + and − signs are convenient symbols for opposite directions. We must show from dynamical considerations that if either force or arm be reversed in direction, then the moment is reversed: which corresponds accurately to the algebraic rule, that if the sign of either factor is reversed the sign of the product is reversed. This is in essence the account of the matter which I have given on p. 180.—I remain, dear Sir, yours faithfully, A. G. CRACKNELL.

In answering Mr. Cracknell's second letter, I will take the points in the order he puts them.

(1) As Mr. Cracknell interprets $+3-7=-4$, it is first of all not an equation but a statement of equivalence of *operations*, and therefore in no way corresponds to the "sum of a set of positive and negative *numbers*" (quoted from ll. 1, 2 on page 173). Even as a statement of equivalence of operations it is not complete, for Mr. Cracknell has to accept numbers less than 4. As I read it from the context quoted above, I made it an equation, *i.e.* a statement that $+3-7$ and -4 stood for the same number; hence I said the first step of $+3-7-6+2$, namely "*take* the number 3 and subtract 7" was impossible.

(2) There is no reason why Mr. Cracknell should "abandon the unfortunate equation in despair." He has all the material for a correct *proof* in his preliminary chapters. He has only to prove that $\frac{a}{b} \times \frac{c}{d}$ works out the same as $\frac{ac}{bd}$, when $\frac{a}{b}$, $\frac{c}{d}$ are *fractional forms denoting integers*, and then define the operation $\times \frac{c}{d}$ by the "equation" (sic) $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. Perhaps it would be better to state it as a convention, explaining that there cannot, for convenience' sake, be *two* laws for fractional forms, one for the case when the form happened to be an integer, the other when it was not: in other words, precisely as he would "find a meaning for a^0, $a^{\frac{1}{2}}$, etc." on the assumption that they obeyed the index laws for positive integral indices. I again put my conundrum, "6 lbs. of sugar $1\frac{1}{2}$ times = ?"

(3) This paragraph simply states in a clearer fashion the point I made, that in calculations with moments the numbers representing the *measures* are alone involved.

(4) I do not quite understand this. Surely Mr. Cracknell does not intend to state that one actually multiplies "a force" by "an arm" to get a moment.

I think in (3) and (4) he emphasizes the wrong word. If he wrote "it is not sufficient to *say* that the + and − signs are convenient symbols for opposite directions, I should agree with him. It must be *shown* that this convention never gives a contradictory result. But once it is shown, that interpretation holds good; the rest of the work, the calculation, is with pure number.

Thus he is quite wrong in saying in the formula he gives on page 179,

height of balloon above the sea $= h + dt$ feet,

that h and dt (on page 180) represent *distances*, d represents a velocity, and t a time; for the symbols h, d, t all represent *numbers*.

But his second letter seems to be quite beside the mark as having any reference to the original criticism. I took exception to his sweeping statement, inferred rather than strictly stated, that because he has shown that +3 is the "reverse" of −3 as a *number* that (+3) and (−3) are "reverses" as *operators*. One might as well say then, that a^{-3} is the "reverse" of a^{+3}, and therefore is a cube root. X. Y. Z.

NAPIER TERCENTENARY CELEBRATION, JULY 1914.

JOHN NAPIER'S *Logarithmorum Canonis Mirifici Descriptio* was published in 1614; and it is proposed to celebrate the tercentenary of this great event in the history of mathematics by a Congress, to be held in Edinburgh on Friday, 24th July, 1914, and following days.

The Celebration is being held under the auspices of the Royal Society of Edinburgh, on whose invitation a General Committee has been formed, representing the Royal Society of London, the Royal Astronomical Society, the Town Council of Edinburgh, the Faculty of Actuaries, the Royal Philosophical Society of Glasgow, the Universities of St. Andrews, Glasgow, Aberdeen, and Edinburgh, the University College of Dundee, and many other bodies and institutions of educational importance.

The President and Council of the Royal Society of Edinburgh have now the honour of giving a general invitation to mathematicians and others interested in this coming Celebration.

The Celebration will be opened on the Friday with an Inaugural Address by Lord of Appeal Sir J. Fletcher Moulton, F.R.S., LL.D. (Edin.), etc., followed by a Reception given by the Right Honourable the Lord Provost, Magistrates and Council of the City of Edinburgh. On the Saturday and Monday the historical and present practice of computation and other developments closely connected with Napier's discoveries and inventions will be discussed.

A Memorial Service will be held in St. Giles' Cathedral on the Sunday.

Among many who have expressed a warm interest in the Celebration, and who hope to take part in the Congress, may be mentioned Professor Andoyer, Paris; Professor J. Bauschinger, Strassburg; Professor Hume Brown, Historiographer Royal for Scotland; Professor F. Cajori, Colorado, U.S.A.; Professor G. A. Gibson, Glasgow; Dr. J. W. L. Glaisher, Cambridge; Professor Lang, St. Andrews; Professor Macdonald, Aberdeen; Professor E. Pascal, Naples; Professor Karl Pearson, London; Professor Eugene

Smith, New York; Professor Steggall, Dundee; Professor Whittaker, Edinburgh.

Merchiston Castle, the residence of Napier, has long been occupied by the well-known public school, which draws pupils from all parts of the British Empire. The Governors of the School have kindly invited the members of the Congress to visit the Castle and Grounds on the Saturday afternoon.

Relics of Napier, collected by Lord Napier and Ettrick and other representatives of the Family, will also be on view; and it is intended to bring together for exhibition books of Tables and forms of Calculating Machines, which may reasonably be regarded as natural developments of the great advance made by Napier.

Individuals, Societies, Universities, Public Libraries, etc., may become Founder Members on payment of a minimum subscription of £2; and each Founder Member will receive a copy of the Memorial Volume, which will contain addresses and papers read before the Congress, and other material of historic and scientific value. It is important to secure as many Founder Members as possible, so that a volume may be brought out worthy of the memory of Napier.

Ordinary Subscribers attending the Celebration may receive copies of the Memorial Volume at a reduced price.

Subscriptions and Donations should be sent to the Honorary Treasurer, Mr. Adam Tait, Royal Bank of Scotland, St. Andrew Square, Edinburgh.

All who are interested in this proposed Celebration are respectfully invited to communicate with the General Secretary of the Royal Society of Edinburgh, 22 George Street, Edinburgh, and to announce their intention of being present.

C. G. KNOTT,
General Secretary, Royal Society of Edinburgh.

January 1914.

NOTICE.

READERS possessing any slide rules of historic interest (or allied instruments), and willing to lend the same for exhibition at the Napier Tercentenary Celebration in July, are requested to communicate with Dr. Stokes, The University, Glasgow.

MATHEMATICAL CONGRESS IN PARIS, 1914.

ITINERARY.

Tuesday, March 31st. Leave London (Victoria Station) at 10 a.m. *via* Newhaven and Dieppe. Arrive Paris (St. Lazare Station) at 6.17 p.m.

Wednesday, April 1st; Thursday, April 2nd; Friday, April 3rd; Saturday, April, 4th. In Paris.

Sunday, April 5th. Leave Paris (St. Lazare Station) at 10 a.m. *via* Newhaven and Dieppe. Arrive London (Victoria Station) at 6.35 p.m.

INCLUSIVE FARE, £4 4s.,

providing:

1. Second-class return ticket from London to Paris *via* Newhaven and Dieppe (first-class travel throughout can be supplied on payment of a supplement of 9s.).

2. Comfortable hotel accommodation, consisting of meat breakfast, table d'hôte dinner, bedroom, lights and attendance, commencing with dinner on Tuesday, March 31st, and terminating with breakfast on Sunday, April 5th.

3. Transfer of passengers and baggage between station and hotel on arrival and departure in Paris.

4. Fees in connection with train reservations.

5. All fees to hotel and railway servants, and to the drivers of the conveyances utilised.

6. Services of a competent Courier throughout, who will generally supervise the arrangements and act as Interpreter whenever necessary.

SHORT SEA ROUTE.

Members can be supplied with travel tickets *via* Folkestone and Boulogne, in which case the fares will be: £4 11s. 3d. second class, or £5 10s. 8d. first class. Second-class passengers may travel first class on the steamers on payment of 4s. extra.

HOTEL ACCOMMODATION.

The fares quoted above cover comfortable second-class accommodation at the Hôtel Mont Fleuri, or similar establishment.

Members may stay at the Hôtel du Louvre, or similar hotel, on payment of a supplement of 16s.; or at the Hôtel Continental, or another house of that class, on payment of £2 5s. over the fare of £4 4s., or £4 11s. 3d. if travelling *via* Folkestone and Boulogne.

Further particulars can be obtained from Thos. Cook & Son, Ludgate Circus, London, E.C.

REVIEWS.

A School Course in Geometry. Including the Elements of Trigonometry and Mensuration and an Introduction to the Methods of Co-ordinate Geometry. By W. J. DOBBS. Pp. 427. 3s. 6d. 1913. (Longmans, Green and Co.)

This is a book which should prove of great interest to all teachers of Mathematics, more so perhaps than the sketch of his ideas given by Mr. Dobbs in a recent *Gazette* would suggest.

The book is a highly original one, and its main feature is that the fundamental propositions are derived from ideas of the rotation, translation, and folding of geometrical figures—the usual tests for congruence of triangles being postponed.

This procedure simplifies the proofs of some propositions, complicates those of others, and makes the order seem very curious in some cases.

Instances are—(i) in proving the equivalence of parallelogram and rectangle a motion of translation brings one of the congruent triangles on to the other: in Euclid's proof of Pythagoras' theorem rotation through a right angle is used for the same purpose.

(ii) the Angle-sum property of a triangle is deduced from the fact that the angles of a parallelogram make four right angles, which is itself proved by sliding one angle on to the corresponding exterior angle.

(iii) that a quadrilateral whose opposite angles are equal is a parallelogram is proved on p. 45, while the corresponding result with "sides" instead of "angles" is not given till p. 232.

A striking feature is the early introduction of the Cosine. This is introduced in Ch. IV., and used to prove Pythagoras' theorem before area has been mentioned. Some will regard this as giving this theorem for perhaps the first time its proper place as one dealing with the squares of (numerical) lengths and not really concerned with areas.

Chapters on Area, Symmetry, Inequalities, and the Non-Symmetrical Properties of the Circle complete the "Geometry to the end of the Circle," after which another striking feature is reached—the chapter on Gradient.

Here Coordinate Axes are introduced and the notation Δx. A very curious

point is that the author uses for preference axes that are not rectangular. The gradient of a line is $\frac{\Delta y}{\Delta x}$, but is not the tangent of the inclination to the axis, and the opportunity to introduce the tangent is not taken.

This will make it difficult to use the book in conjunction with an Algebra which introduces the tangent at an early stage as the gradient of a graph.

The next four chapters carry Trigonometry to the solution of Triangles. Tan, Cot, Sec, Cosec are still postponed, but in Chap. XII. we at last come to the Sine [how the Author must regret the old notation: if only it were the "Cosco"!].

Then we have two belated chapters, Duplicate and Similar Triangles, and Similar Figures, which might have been placed a hundred pages earlier.

"Duplicate," instead of "Congruent," seems to serve no purpose. Other such changes are "radial" for "polar" coordinates [no doubt in pity for the poor overworked "polar"], "line-segment" for line, and "one and only parallel" for parallel: also "protract" an angle.

The next three chapters finish the course of Trigonometry, with which much Analytical Geometry has been mixed, and the book is completed by two chapters on Analytical Geometry, and two on gradients and areas of curves which involve the elements of Differential and Integral Calculus.

At the end of each chapter there are from 20 to 80 examples (the total number is just under 1000). No examples are given except at the ends of chapters, and there are few specimen worked examples.

It will be seen that the book contains a great deal under one cover. It should be added that it is well printed, and arranged so that reference is fairly easy: and that the style is throughout interesting, homogeneous, and full of conviction.

C. O. TUCKEY.

Entwickelung der Mengenlehre und ihrer Anwendungen. Umarbeitung des im VIII. Bande der **Jahresberichte der Deutschen Mathematiker-Vereinigung** erstatteten Berichts. Gemeinsam mit Hans Hahn herausgegeben von Artur Schoenflies. Erste Hälfte: Allgemeine Theorie der unendlichen Mengen und Theorie der Punktmengen. Von ARTUR SCHOENFLIES. Pp. xi, 389. Paper covers, 16 marks; cloth, 18 marks. 1913. (Leipzig and Berlin: B. G. Teubner.)

This is a second edition of the first part (1900) of Prof. Schoenflies' well-known Report on the Theory of Aggregates. It contains many of the additions made in the second part (1908) of this Report, but, as a rule, this edition and the second part in question are occupied with different parts of the subject. Thus, that part of Hausdorff's work on linear types which was described in 1908 is not considered here (p. v). This new edition is subdivided into two parts; and the second part, which treats of the theory of real functions, will be published this summer under the editorship of Hans Hahn.

Like the first edition, this Report is a mixture of history and text-book (p. iv). Unlike the first edition, some space is devoted to the consideration of *finite* numbers (pp. v, 89-91; but cf. the old view of p. 4). As in the first edition, the paradoxes and the relations of the theory of aggregates to logic are to be disregarded (pp. v-vi), but Prof. Schoenflies cannot resist mentioning with approval his "solution" (pp. vi, 7, 98), where, unaffected by logic, he excommunicated from the Holy Church of mathematics a "something which cannot be the object of logical consideration according to the law of contradiction." Some will remember with amusement the eloquence—logic was of course not adequate—used by Prof. Schoenflies to banish this awkward and irreverent ghost to the shadowy domains of philosophy.

Although the multiplicative axiom is—as it was not in the first edition—mentioned (pp. 16, 174-176), yet—much as in the first edition—it is not mentioned when it is used on pp. 20-21, 25, 41 (note 2), 110, 132, 221-222, 235, 236, 240-241, 254. And, on this subject, there is no mention of the fact that the comparability of cardinal numbers is equivalent to the supposition of the above axiom (or to the supposition that every aggregate can be well-ordered). Here we may refer to the welcome addition of the first note on p. 111.

Other mistakes do not seem to be very numerous. These are some: the second reference in the third note on p. 136 is wrong; the present reviewer has proved

that Veblen's theorem mentioned in the last note on p. 238 is subject to exceptions; some of Whitehead's work should have been mentioned on p. 13; there is certainly an error on p. 232: the existence of a limes-point cannot be inferred on empirical grounds; and it is not pointed out that Bernstein's remark (p. 44) is irrelevant.

We turn to the many additions to this edition. Whitehead's work on cardinal arithmetic is referred to on pp. 10, 15-16; later work on the Schröder-Bernstein theorem is described (pp. 36, 40), as also is J. König's theorem and its generalizations (p. 65). As regards ordinal types, we have accounts of work of Hausdorff (pp. 77, 185), Hessenberg and Jacobsthal (p. 149), the theory of multiply-ordered aggregates (p. 84), the theory of number-classes in general (p. 120), the question of well-ordering (p. 170). In the theory of point-aggregates, we have additions on the theorem of Heine and Borel (p. 234), the work of Haar and D. König (pp. 265, 280), and Lebesgue's conception of measurability (p. 313, 368).

It is a pity that the relations of the theory of aggregates to logic are not considered; otherwise the frequent mistake of mathematicians like Prof. Schoenflies (cf. p. 133) that, if $a \geqq b$ and $b \geqq a$, then $a = b$ follows, not by logical principles alone, but by the theorem of Schröder and Bernstein, might have been avoided.

This edition is a very useful piece of work. It is not thoroughly reliable nor well-digested, and seems very inferior to what the first edition was in 1900. Still, it cannot be neglected, as it is—though narrow in view—a serious attempt to deal with a very important and fundamental part of mathematics.

Veränderliche und Funktion. By MORITZ PASCH. Pp. vi, 186. Paper, 6 marks; cloth, 7 marks. 1914. (Leipzig and Berlin: B. G. Teubner.)

This work is a sequel to Prof. Pasch's *Grundlagen der Analysis* of 1909, and there are frequent references to this earlier work. "In the course of investigation of the conception of a function," says the author on p. iii, "I saw that it was not permissible to distinguish sharply between function and expression (*Ausdruck*)." An "expression" is (p. 11) a compound name. Thus, here again we see that nominalistic tendency which made the *Grundlagen*, in its definition of irrational numbers, so far inferior to the *Einleitung in die Differential- und Integralrechnung* of 1882. In the earlier work, Prof. Pasch defined real numbers as "segments" of rationals, exactly as in the modern (Russell's) manner. The more historical remarks on Kronecker's arithmetical tendency (pp. 21, 29, 36-38, 156-157), Ampère's "proof" of the existence of a derivative, and so on (pp. 122-129, 29-30), Cauchy and uniform continuity (pp. 129-131), Dirichlet's principle (pp. 131-133), and Hesse's form (pp. 134-136) are very valuable.

The four last examples are to illustrate the progress of mathematical thought. Also useful remarks on the earlier work of the author and others on the theory of real functions and variables are given on pp. 139-142. There is an appendix, derived from an earlier publication of 1904, on the introduction of imaginaries (pp. 27, 158-181).

It seems to the reviewer that this work is chiefly interesting from the point of view of the principles. The need for rigid system has become more and more evident of late years, and even mathematicians feel bound to bestow at least a brief recognition on "the class of all classes" (p. 40). PHILIP E. B. JOURDAIN.

The Theory of Relativity. By ROBERT D. CARMICHAEL. (Mathematical Monographs, edited by Mansfield Merriman and Robert S. Woodward, No. 12). Pp. 74. $1.00. 1913. (New York: Wiley & Sons. London: Chapman & Hall).

It is now more than eight years since the theory of relativity was expounded by Einstein, and, although the literature of the subject is already considerable, this is practically the first presentation in book form which has been offered to English readers. The author is a mathematician, and has confined his exposition to what may be called the external aspects of the theory, to the exclusion of molecular physics. In this way he has presented the essential characteristics of the theory without assuming on the part of the reader anything but the most elementary notions of geometry and dynamics. The postulates are clearly stated, and the results are developed by a modification of the methods of Lewis and Tolman.

This simple and logical account will serve a useful purpose by showing what assumptions we are in the habit of making, and wherein these admit of modification without contradicting the evidence of our senses.

Das Relativatsprinzip: eine Sammlung von Abhandlungen. H. A. LORENTZ, A. EINSTEIN, H MINKOWSKI. (Fortschritte der mathematischen Wissenschaften in Monographien, herausgegeben von Otto Blumenthal, Heft 2.) Mit Anmerkungen von A. Sommerfeld, und Vorwort von O. Blumenthal. Pp. 89. M 3.00. 1913. (Leipzig: Teubner.)

The firm of Teubner has collected and printed, under the editorship of Prof. Blumenthal, this useful volume of original memoirs on the theory of relativity. The pamphlet, which bears as frontispiece an excellent portrait of the late Hermann Minkowski, contains Einstein's classical paper "Zur Elektrodynamik bewegter Körper," together with some of Lorentz' contributions, and the celebrated address of Minkowski, "Raum und Zeit." The reprint of Minkowski's suggestive article will be especially welcomed as the paper itself has been for some time out of print and accessible only in his collected works. D. M. Y. S.

Slide-Rule Notes. By H. C. DUNLOP and C. S. JACKSON. Pp. 127. 2s. 6d. net. 1913. (Longmans, Green and Co.)

As might be expected when one of the authors is known to many in the Services as "Slide-Rule Jackson," this little book contains in an admirable form a full account of the theory and use of the type of slide-rule in common use. The explanations are good and sufficient, and are made still more clear by the use of diagrams showing the rule in outline which indicate only those graduations which are actually required for the particular problem considered—a plan which, though strange at first sight, has much to commend it for a learner. Another feature which often recurs is the system of giving a mnemonic for the operations required for certain processes, although it must be confessed that the use of the mnemonic usually involves considerable juggling to make the problem fit in with it, and it therefore seems scarcely worth the trouble.

Even the expert will find much of interest in the book, such as the methods of solving quadratic and cubic equations with the rule, whilst the miscellaneous notes on the equations to the scales and on the ways of dealing with some complex operations will be new to most: the teacher will find most useful the numerical examples which are included in each section. Perhaps one small point may be mentioned by way of criticism: all numbers and algebraic quantities are printed in heavy type, which makes the reading very confusing, as the subheadings are not noticed on looking through the pages, and it is quite difficult to find the examples. If this mode of printing is to be retained it might be better to put the subheadings in the margin: the use of the book for purposes of reference would be easier. F. J. DYKES.

The Calculus and its Applications. By R. G. BLAIN. 4s. 6d. net. Second Impression. 4s. 6d. net. 1911. (Constable.)

This clearly printed and well-written little introduction to the subject and its applications has apparently found among the engineering students to whom it especially appeals the welcome which is due to its sterling merits. There are still one or two unimportant uncorrected slips, *e.g.* l. 12, p. 137, for $\frac{I}{R}$ read $\frac{1}{R}$; B is missing from the diagram, p. 9.

An Algebra for Preparatory Schools. By TREVOR DENNIS. Pp. viii + 155. 2s. 1913. (Cambridge University Press.)

This little book follows the syllabus for Public School non-specialist boys, and develops the subject entirely by examples, a large number of which are intended for oral use. "Perhaps the strongest impression received from teaching young boys is that they forget as quickly as they learn. For this reason constant revision is introduced at every stage." It seems admirably adapted for its purpose.

Annuaire pour l'An 1914, publié par le Bureau des Longitudes. Avec des Notices scientifiques. Pp. vii + 502 + 9 + 107 + 34 + 42. 1 fr. 50 c. 1914. (Gauthier-Villars.)

This year the *Annuaire* confines its tables to those of physical and chemical data, to seismology and solar physics. Geographical and statistical tables will be included in next year's volume, with whatever astronomical tables are omitted in that for the present year. M. Andoyer has rewritten the sections dealing with the fundamental astronomical ideas, as well as the tables in connection therewith, and the data relative to the satellites. M. Rocques Desvallées deals with the calendars of civilised races, and M. Puiseux with the physical constitution of the moon. MM. Bigourdan and Deslandres write respectively on seismology and solar physics, and M. de Gramont is responsible for the sections on stellar spectra. The Notices this year are three in number: La Déformation des Images dans les Lunettes, M. Matt; Le jour et ses Divisions, M. G. Bigourdan; Sur le 17e Conférence générale de l'Association géodésique internationale, M. Baillaud.

Examples and Test Papers in Algebra. W. J. Walker. Part I. (including Quadratic Equations in one Unknown). Pp. vii + 162 + 32. 1s. 3d.; with answers, 1s. 6d. 1913. (Mills & Boon.)

The teacher is left to supply definitions and explanations, and the sets of examples are intended to lead the student to formulate his own rules. The problems for graphical solution need no "ingenious or laborious devices." There are plenty of mental or oral questions, and fifty test papers. The type is large, and most left-hand pages begin with a new exercise. The examples are for the most part original, and are carefully arranged.

Elementary Graphic Statics. By J. T. Wight. Pp. xii + 227. 3s. 6d. 1913. (Whittaker.)

This admirable little volume is cordially to be recommended. Although we might indicate certain omissions the inclusion of which would have enhanced its value, yet it may fairly be said that somewhere or other within its covers *all* the standard problems and exercises have been either demonstrated or indicated. There are few graphical problems that a student will not have a good chance of tackling successfully if he has mastered the contents of this little book. More particularly will it be found useful for the student who wishes to supplement what he has been learned in the lecture-room or the exercises of the drawing office. The diagrams are small but clear. It is, however, a disadvantage that here and there they refer to letterpress over the page. In the Introduction the author writes: "Note carefully that the scale of the diagram must be given alongside, otherwise the diagram is useless." Such scales seem to be missing from many of the force diagrams in the book.

BOOKS, ETC., RECEIVED.

School Science and Mathematics. Edited by C. H. Smith. Vol. XIII. No. 8, Nov. 1913. 2$ per ann. (Smith & Turton, Mount Morris, I.M.)

A Thread of Mathematical History and some Lessons. R. D. Carmichael.

Das Relativitätsprinzip. By H. A. Lorenz, A. Einstein, and H. Minkowski. Pp. 88. 3m. 1913. (Teubner.)

Das Problem der Kreisteilung. By A. Mitzscherling, with Preface by H. Liebmann. Pp. vi + 214. 9m. 1913. (Teubner.)

Analytische Geometrie der kubischen Kegelschnitte. By O. Staude. Pp. viii + 242. 9m. 1913. (Teubner.)

Betrachtungen über mathematische Erziehung vom Kindergarten bis zur Universität. By B. Branford. Edited by R. Schimmack and H. Weinreich. Pp. viii + 403. 6m. 1913. (Teubner.)

Das Foucaultsche Pendel und die Theorie der Relativenbewegung. By A. Denizot. Pp. 76. 3m. 1913. (Teubner.)

Geheimnisse der Rechenkünstler. By P. MAENNCHEN. (No. 13. Mathematische Bibliothek, edited by W. LIETZMANN and A. WITTING.) Pp. 48. 80 pf. 1913. (Teubner.)

Beispiele zur Geschichte der Mathematik. (No. 15. Math. Bibliothek.) Part II. By A. WITTING and M. GEBHARDT. Pp. viii + 61. 80 pf. 1913. (Teubner.)

Grundzüge der mathematisch-physikalischen Akustik. By A. KALÄHNE. Part II. Pp. x + 225. 5 m. 40 pf. or 6 m. bd. 1913. (Teubner.)

Annals of Mathematics. Edited by O. STONE and others. Ser. II. Vol. 15. No. 2. Dec. 1913. 2$ per vol. (Lancaster, Pa., and Princeton, N.J.)

On the numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$. R. D. CARMICHAEL. *Geometric Characterisation of Isogonal Trajectories on a Surface.* J. LIPKA. *On the Second Variation, Jacobi's Equation and Jacobi's Theorem for the Integral* $\int F(x, y, x', y')dt$. A. DRESDEN. *Quartic Surfaces Invariant under Periodic Transformations.* F. R. SHARPE and F. M. MORGAN. *Postulate Sets for Abelian Groups and Fields.* W. A. HURWITZ.

Tôkyô Sûgaku-Buturigakkwai Kizi. (Proc. Tôkyô Math. Phys. Soc.) Sept. 1913.

The American Mathematical Monthly. Edited by H. E. SLAUGHT and others. Vol. XX. No. 9. Nov. 1913. 2$ 50 c. per ann. in advance.

Number Systems of the N. American Indians. W. G. EELLS. *Synthetic Projective Geometry as an Undergraduate Study.* W. H. BUSSEY. *A Note on the Solution of Linear Differential Equations.* C. R. MACINNES. *A Graphical Solution of the Differential Equation of the First Order.* T. R. RUNNING.

Models to Illustrate the Foundations of Mathematics. By C. ELLIOTT. Pp. viii + 116. 2s. 6d net. 1913. (Lindsay, Edinburgh.)

Analytic Geometry and Principles of Algebra. By A. ZIWET and L. A. HOPKINS. Pp. viii + 369. 7s. net. 1913. (Macmillan.)

Plain and Solid Geometry. By W. B. FORD and C. AMMERMAN. Pp. ix + 321 + xxxiii. 5s. 6d. net. 1913. (Macmillan.)

Exercises in Mathematics. By D. B. MAIR. Pp. xi + 469. 4s. 6d. with Answers and Hints; 3s. 6d. without. 1914. (Macmillan.)

Slide-Rule Notes. By Col. H. C. DUNLOP and C. S. JACKSON. Pp. 127. 2s. 6d. net. (Longmans, Green.)

A School Course in Geometry, including the Elements of Trigonometry and Mensuration, and an Introduction to the Methods of Co-ordinate Geometry. By W. J. DOBBS. Pp. xxii + 427. 3s. 6d. (Longmans, Green.)

Bolletino della "Mathesis." Num. 3. Anno V. Dec. 1913. (Manuzio, Rome.)

Sulle Trasformazioni Voigt-Lorentz in Elettrodinamica. Nota di Critica Scientifica con Prefazione. By A. DEL RE. (Tip. della R. Accad., Naples.)

Bulletin of the American Mathematical Society. Edited by F. N. COLE and others. Vol. XX. No. 3. Dec. 1913.

The Madison Colloquium. A. DRESDEN. *The Vienna Meeting of the Deutsche Mathematiker-Vereinigung.* V. SNYDER. *On Binary Modular Groups and their Invariants.* L. E. DICKSON. *On some Systems of Collineation Groups.* H. H. MITCHELL. *On the Summability of Fourier's Series.* T. H. GRONWALL. *Note on Pierpont's Theory of Functions.* G. A. BLISS.

Solid Geometry. By W. B. FORD and C. AMMERMAN. Pp. ix + 215-321 + xlix. 3s. 6d. net. 1913. (Macmillan.)

Colorado College Publication. General Series. Nos. 57, 58, 59, 60, 63, 64, 66. Published by Board of Trustees of Colorado College, Colorado Springs, U.S.A.

Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree. By FELIX KLEIN. Translated by G. G. MORRICE. 2nd and revised edition. Pp. xvi + 289. 10s. 6d. net. n.d. (Kegan Paul.)

American Journal of Mathematics. Edited by F. MORLEY. Vol. XXXVI. No. 1. 5$ per annum. Jan. 1914. (Johns Hopkins Press, Baltimore.)

Determination of all primitive Collineation Groups in more than four Variables which contain Homologies. H. H. MITCHELL. *On non-homogeneous Equations with an infinite Number of Variables.* R. D. CARMICHAEL. *On constrained Motion.* P. FIELD. *Covariant Curves of the plane rational Quintic.* J. I. TRACEY. *The Group of Isomorphisms of an Abelian Group and some of its Abelian Sub-groups.* G. A. MILLER. *Self-Projective rational Curves of the fourth and Fifth Orders.* R. M. WINGER. *Iterated Limits in general Analysis.* R. E. ROOT.

Introduction géométrique à quelques théories physiques. By E. BOREL. Pp. vii + 137. 5 frcs. 1914. (Gauthier-Villars.)

Cours de Mécanique professé à l'Ecole Polytechnique. By L. Lecornu. Vol. I. Pp. vii+538. 18 frcs. 1914. (Gauthier-Villars.)

Etoiles magiques à 8, 16, *et* 20 *branches* (24, 64, *et* 100 *points*) *et Rosaces Hypermagiques* (16, 25, *et* 36 *points*). By C. Salomon. Pp. 21. 1 frc. 50. 1913. (Gauthier-Villars.)

Etoiles magiques à 10 *et* 12 *branches* (30, 36, *et* 48 *points*) *et Hexagones et Octogones magiques.* By C. Salomon. Pp. 26. 1 frc. 50. 1913. (Gauthier-Villars.)

Revista de la Sociedad Matematica Española. Jan.-Dec., 1913.

Intermediate Mechanics. By F. C. Turner and J. M. Bose. Pp. xii+332. 4s. 6d. 1914. (Longmans, Green.)

Practical Mathematics for Technical Students. Part I. By T. S. Usherwood and C. J. A. Trimble. Pp. 370. 3s. 6d. 1914. (Macmillan.)

A Text-Book on Spherical Trigonometry. By R. E. Moritz. Pp. vi+67. 4s. 6d. net. 1913. (Chapman & Hall.)

Plane and Spherical Trigonometry (with Five-Place Tables). By R. E. Moritz. Pp. xvi+357+96. 10s. 6d. net. 1913. (Chapman & Hall.)

The Algebra of Logic. By L. Couturat. Pp. xiv+98. 1914. (Open Court Co.)

Examples and Test Papers in Algebra. By W. J. Walker. Pp. x+162+32. 1s. 6d. 1913. (Mills & Boon.)

The Groundwork of Arithmetic. A Handbook for Teachers. By Miss M. Punnett. Pp. xii+234. 3s. 6d. 1914. (Longmans, Green.)

Exercises in the Groundwork of Arithmetic. By Miss M. Punnett. Bk. i. pp. 58, 4d.; Bk. ii. pp. 107, 6d.; Bk. iii. pp. 84, 6d. 1914. (Longmans, Green.)

The Principle of Relativity in the Light of the Philosophy of Science. By Paul Carus. With an Appendix containing a Letter from the Rev. James Bradley on the Motion of the Fixed Stars. 1727. Pp. 105. $1 net. 1913. (Open Court Pub. Co.)

The Mechanistic Principle and the Non-Mechanical. An Inquiry into Fundamentals, with Extracts from Representations of either Side. By Paul Carus. Pp. 125. $1 net. 1913. (Open Court Pub. Co.)

School Science and Mathematics. Vol. XIII. No. 9. Dec. 1913. Edited by C. H. Smith. 25 c. 1913. (Smith & Turton, Chicago.)

Economics in the Course of High School Mathematics. F. O. Hester. *Organisation of Vocational Secondary Mathematical Courses.* W. T. Stratton. *Primes in A.P.* A. Martin. *Factoring* ax^2+bx+c. J. M. Gallagher. *Proof of Theorem of Pythagoras.* H. Hartig.

Gazeta Matematica. June-Dec. 1913. 7.50 lei per ann. (Göbl, Bucharest.)

Annaes scientificos da Academia Polytechnica do Porto. Edited by F. G. Teixeira. Vol. VIII. No. 3. 6 frcs. per ann. 1913.

Sur l'interpolation. P. Martinotti. *Sur une loi de force centrale.* E. Turrière. *Sur les biquadratiques gauches de première espèce.* C. Servais.

Proceedings of the Tôkyô Mathematico-Physical Society. Oct. 1913. Ser. 2. Vol. VII. No. 9.

On the Formula for the Arc of a Circle in the "Kwatuyō Sampō," and allied subjects. Y. Mikami

Exercises from "A New Algebra." Parts I.–IV. By S. Barnard and J. M. Child. Pp. 274. 2s. 6d. 1913. (Macmillan.)

Marsh's Mathematics Work-Book. Designed by H. W. Marsh. 3s. net. 1913. (Chapman & Hall, Wiley.)

Constructive Text-Book of Practical Mathematics. By H. W. Marsh. Vol. II. Technical Algebra. Part I. First edition. First thousand. 8s. 6d. net. 1913. (Chapman & Hall, Wiley.)

"Squaring the Circle": A History of the Problem. By E. W. Hobson, F.R.S. Pp. 57. 3s. net. 1913. (Camb. Univ. Press.)

Examples in Physics. By H. Freeman and E. Jobling. Pp. 96. 1s. net. 1913. (Heffer, Cambridge.)

A School Statics. By G. W. BREWSTER and C. J. L. WAGSTAFF. Pp. viii+248. 1913. (Heffer, Cambridge.)

Elementary Graphic Statics. By J. T. WIGHT. Pp. xii+227. 4s. net. 1913. (Whittaker.)

Solutions des Exercices proposés dans Les Eléments de Mathématiques supérieures. By H. VOGT. Pp. 277. 6 frcs. 1913. (Vuibert.)

Entwickelung der Mengenlehre und ihre Anwendungen. By A. SCHOENFLIES. Pp. xi+389. 16 m. (18 m. bd.) 1913. (Teubner.)

Veränderliche und Funktion. By M. PASCH. Pp. vi+186. 6 m. (7 m. bd.) 1913. (Teubner.)

Junior Geometry. By A. G. CRACKNELL. Pp. viii+276. 2s. 6d. 1913. (Univ. Tutorial Press.)

Wiskundig Tijdschrift. Edited by F. J. Vaes. Jan.-Dec. 1913. (Visser, Haarlem.)

THE LIBRARY.

THE Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2 copies of *Gazette* No. 2 (very important).
1 or 2 „ „ No. 8.
2 or 3 copies of Annual Report No. 11 (very important).
1 or 2 „ „ Nos. 10, 12 (very important).
1 copy „ Nos. 1, 2.

FOR SALE.

A Junior Trigonometry. By W. G. BORCHARDT and A. D. PERROTT. Pp. vii+220+xvii+xx. 3s. 6d. 1914. (Bell & Sons.)

Richtige elementare Lösung des Fermat'schen Problems $x^n+y^n=z^n$. By H. MAURER. Pp. 15. 1 fr. 1914. (Orell Füssli, Zürich.)

Mathematical Gazette. Quarto No. 6. Out of print and very scarce. Offers to X.Y., c/o Editor.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.